SWAT 95-96/121
PRELIMINARY

Exact quantum S-matrices for solitons in simply-laced affine Toda field theories

Peter R. Johnson[1]

Department of Physics,
University of Wales at Swansea,
Singleton Park,
Swansea,
SA2 8PP, Wales, UK.

Abstract

We obtain exact solutions to the quantum S-matrices for solitons in simply-laced affine Toda field theories, except for certain factors of simple type which remain undetermined in some cases. These are found by postulating solutions which are consistent with the semi-classical limit, $\hbar \to 0$, and the known time delays for a classical two soliton interaction. This is done by a 'q-deformation' procedure, to move from the classical time delay to the exact S-matrix, by inserting a special function called the 'regularised' quantum dilogarithm, which only holds when $|q| = 1$. It is then checked that the solutions satisfy the crossing, unitarity and bootstrap constraints of S-matrix theory. These properties essentially follow from analogous properties satisfied by the classical time delay. Furthermore, the lowest mass breather S-matrices are computed by the bootstrap, and it is shown that these agree with the particle S-matrices known already in the affine Toda field theories, in all simply-laced cases.

MSC Classification codes: 81Q20, 81R50, 81U20, 58F07

Keywords: completely integrable systems, solitons, affine Toda field theory, S-matrix, quantum theory, quantum groups

[1]current address: CEA-Saclay, Service de Physique Théorique, 91191 Gif-sur-Yvette Cedex, France

1 Introduction

The affine Toda field theories are integrable systems, depending on a Lie algebra g, which admit soliton solutions interpolating the degenerate vacua in the potential, provided the coupling constant β in the model is chosen to be purely imaginary. In the simply-laced cases, the classical single soliton solutions can be grouped together into species or equivalently associated with a node of the Dynkin diagram of g. All the solitons of a particular species have the same mass, but different topological charges. However there can be species with the same mass. The topological charges of these single solitons of a particular species are contained in the weights of the fundamental representation associated with that node. Furthermore, the fusing rule $i + j \to k$ (Dorey's fusing rule [1]), when two solitons of species i and j fuse into a third single soliton of species k, is always contained within the Clebsch-Gordon decomposition of the tensor product of the two fundamental representations V^i and V^j. These facts suggest that the S-matrix for the collision of two solitons of species i and j in the quantum theory is given by the R-matrix of some affine quantum group (up to a multiplicative scalar factor) intertwining the tensor product of the two fundamental representations V^i and V^j. This means that the S-matrix must satisfy the quantum Yang-Baxter equation with spectral parameter – a necessary property of an S-matrix in an integrable theory, since the infinite number of conservation laws suggest that an n-particle S-matrix will factorise into products of the two-particle result. This can be done in two different ways for the 3-soliton case, implying the quantum Yang-Baxter equation.

This R-matrix approach to finding the S-matrix was first discussed in [2, 3] for the A_n case, and there is also a discussion in [4]. The result for the scalar factor in the sine-Gordon case ($g = su(2)$) is the well known Zamolodchikov-Zamolodchikov solution [5], which is given as an infinite product of gamma functions, or a double infinite product (7.10). We can easily pass between these two descriptions by using the Weierstrass product formula for the gamma function. There is the result, also for sine-Gordon, due to Karowski, Thun, Truong and Weisz [6], given as a neater integral formula (7.9), which, on expansion, can be shown to be the same as the formula given by the Zamolodchikov's.

The purpose of this paper is to announce solutions to the scalar factors which are built out of a special function $S_{q^{-h}}(w)$ called the 'regularised' quantum dilogarithm (6.4). This special function has recently been introduced by Faddeev [7]. For sine-Gordon, we can easily reproduce the Karowski formula [6]. These solutions have only so far appeared in [8], which has been circulated privately. It is not our intention to study the quantum groups and R-matrices, or their relation with quantum integrable systems, in any great detail. We shall take the R-matrices that we shall need from the literature.

There are two advantages in writing the scalar factors in this way. The first is that the semi-classical limit $\hbar \to 0$, can be taken very efficiently. The overall effect of a classical two soliton interaction is a time delay or phase shift. In [9], it is shown that the time delays for simply-laced affine Toda field theories have a vertex operator origin, and are given by $(2h/|\beta|^2)\log X^{jk}(\theta)$, where $X^{jk}(\theta)$ arises from the normal ordering of two vertex operators. It is well known [10], that the S-matrix $S^{jk}(\theta)$, must have the semi-classical limit when $\hbar \to 0$, for Re $(\theta) > 0$,

$$S^{jk}(\theta)\Big|_{\hbar\to 0} \sim \exp\Big(\frac{2hi}{\hbar|\beta|^2}\int_0^\theta d\theta' \log X^{jk}(\theta')\Big). \tag{1.1}$$

Considerable manipulations are required to check this in the formalism involving infinite

products of gamma functions [5, 2], and also for the integral formula due to Karowski *et al* [6], even for sine-Gordon.

This way of writing the solution means that new solutions can be obtained by extrapolating back from the classical $X^{ij}(\theta)$, in cases where solutions are not yet known (the D and E series of algebras). However, we do require input from the R-matrices, which are known for a few of the fundamental representations, in these cases, as discussed in section 11. The method in this paper also provides an alternative description of the A_n theories [2], where the crossing, unitarity, and bootstrap constraints are clearly demonstrated. The extrapolation can be thought of as 'q-deforming' $X^{ij}(\theta)$ to some $X_q^{ij}(\theta)$, by the insertion of appropriate quantum dilogarithms, and then the S-matrix is given by

$$S^{ij}(\theta) = \frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)},$$

modulo the simple factors needed for the different topological charges, which are provided by the R-matrix.

The second advantage is that the crossing, unitarity and bootstrap properties of the S-matrix, which have to be checked, essentially follow from the analogous properties satisfied by the classical $X^{ij}(\theta)$. Again considerable manipulations with the infinite products of gamma functions, which are avoided in the formalism presented in this paper, have to be made in order to check these properties, in the other formalism[5, 2]. Because of these two simplifications it is possible to essentially write down the general solution for an arbitrary simply-laced algebra, modulo some special factors, and to check the semi-classical limit, and the crossing, unitarity, and bootstrap conditions in a general uniform manner.

However it should be noted that the quantum dilogarithm itself, $S_{q^{-h}}(w)$, can be written as an infinite product of gamma functions (6.10), or alternatively as a double infinite product (6.9), and this is how the pole structure of the proposed solutions is studied. If we wanted to, this would allow us to make contact with the previous solutions known in the A_n case.

We shall also show in section 14, that the lowest mass breather S-matrices are the same as the Toda particle S-matrices[11, 12], for the real coupling Toda theories, after the analytic continuation in the coupling $\beta \to i\beta$. This is to be expected if the solutions found are indeed correct, and if we interpret the lowest mass breather b_i, made up of a bound state of a soliton with species i and the anti-soliton $\bar{\imath}$, with the particle i of the quantum field theory (which is still present in the imaginary coupling theories). This is an important independent check on the solutions. However it is remarkable to the author that the result survives the analytic continuation $\beta \to i\beta$. This result was known for the $su(2)$ case in [13], and Gandenberger studied the $su(3)$ case [14], using Hollowood's formula [2]. Here we will employ the general formula for the particle S-matrix (14.2) due to Dorey [12, 15], which holds for an arbitrary simply-laced Lie algebra g.

Contents:

2 The affine Toda field theories and their classical soliton solutions

The affine Toda field theories are based on a Lie algebra g, and have a non-linear interaction term built from the root system of g. Let $\alpha_i : \ i = 1, \ldots, r$, where r is the rank of g, be the simple roots, and let $\alpha_0 = -\psi$, where ψ is the highest root. Let h be the Coxeter number of g. Expand ψ in terms of the simple roots

$$\frac{\psi}{\psi^2} = \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2},$$

defining m_i, $i = 1, \ldots r$, which are positive integers. Define $m_0 = 1$, and then $\sum_{i=0}^{r} m_i \frac{\alpha_i}{\alpha_i^2} = 0$. Now the affine Toda field theories, for an r-component scalar field u, have equations of motion

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{4\mu^2}{\beta} \sum_{i=0}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i . u} = 0. \tag{2.1}$$

Observe that any constant u such that $e^{\beta u.\alpha_i} = 1$ or alternatively $u \in \frac{2\pi}{|\beta|} \Lambda_W(g^\vee)$, the weight lattice of the co-root algebra, is a solution, provided β is chosen to be purely imaginary. Thus we expect soliton solutions to exist which smoothly interpolate these vacua at $x \to \pm\infty$. Hollowood found some exact soliton solutions in the A_{n-1} case [16], using the Hirota method. There is also a discussion of these solutions, and excited modes around them, discovered using the inverse scattering method in [17].

For λ_i a fundamental weight, defined by $\frac{2\lambda_i \cdot \alpha_j}{\alpha_j^2} = \delta_{ij}$, Hollowood found in [16], for a constant $Q \in \mathcal{C}$,

$$e^{-\beta \lambda_i \cdot u} = \frac{1 + Q\omega^{ij} W}{1 + QW}, \quad \omega = e^{\frac{2\pi i}{n}}, \tag{2.2}$$

where $W = e^{2\mu \sin(\frac{\pi j}{n})(e^{-\theta} x_+ - e^{\theta} x_-)}$, with $x_\pm = t \pm x$.

The integer j denotes the *species* of soliton, which can be associated with the fundamental representation V^j of g, which is defined to have highest weight λ_j. Indeed, it has been checked explicitly in the A_{n-1} case [18] that the topological charges of a soliton of species j are weights of V^j. The topological charge is defined to be the difference between u as $x \to \infty$

and $x \to -\infty$, and the charge of (2.2) jumps to different values as we vary the phase of Q. However there still remains the problem that, for $n \geq 4$, not all the weights are filled by classical single soliton solutions, and this begs the question whether there are some missing classical solitons. In what follows we must assume that these missing classical solutions actually exist, or that states are created in the quantum theory, so that a multiplet of solitons of species j transform in the fundamental representation V^j. Otherwise the representations would have to be restricted to physical spaces, to fit in with what is known classically, and somehow incorporated into the residues at the poles in the S-matrix, and the unitarity sum, $S^{ij}(\theta)S^{ji}(-\theta)$, of the S-matrix. These spaces would no longer be representations.

Soliton solutions for the other simply-laced cases were found by Olive, Turok and Underwood [19, 20]. Again there is the problem that there are many missing charges, but the charges of a given species are still contained in the appropriate fundamental representation, as in the A_n case. Within their abstract formalism, the true origin of the different sorts of species of soliton, each species being associated with a node of the Dynkin diagram of g, was discovered. It was shown that the solitons of a given species have the same mass, and crucially the solitons of species i have the same mass as the particle associated with the i^{th} node of the Dynkin diagram in the quantum field theory, up to some global renormalisation of the particle masses. Furthermore, a certain coefficient, $X^{ij}(\theta)$, which we shall call the interaction function, and which plays an important rôle in the multi-soliton solutions, arose from the normal ordering of two vertex operators $F^i(z)$ and $F^j(w)$. Here θ is the relative rapidity of the two solitons of species i and j.

$$F^i(z)F^j(w) = X^{ij}(\theta) : F^i(z)F^j(w) : .$$

Each vertex operator $F^i(z)$ 'creates' a soliton in the formalism. An explicit formula is given for $X^{jk}(\theta)$:

$$X^{jk}(\theta) = \prod_{p=1}^{h} \Big(1 - e^{\theta} e^{\frac{\pi i}{h}(2p + \frac{c(j)-c(k)}{2})}\Big)^{\gamma_j \cdot \sigma^p \gamma_k}. \tag{2.3}$$

Here $c(i) = \pm 1$ is the 'colour' of the node i of the Dynkin diagram of g, which can be bi-coloured in a certain way. σ is a special element of the Weyl group known as the Coxeter element, $\sigma^h = 1$, and $\gamma_i = c(i)\alpha_i$. The Coxeter element σ partitions the root system into r orbits of h elements, with γ_i a representative of each orbit, for $i = 1, \ldots, r$. Also $\gamma_j \cdot \sigma^p \gamma_k = \pm 2, \pm 1, 0$. These integers are known explicitly for each algebra g, and are summarised in tables later on in this paper.

As discussed in [20], $X^{jk}(\theta)$ has poles at some purely imaginary values of θ. These occur if $\gamma_j \cdot \sigma^p \gamma_k = -2$, or if $\gamma_j \cdot \sigma^p \gamma_k = -1$. The latter case is equivalent to $\gamma_j + \sigma^p \gamma_k = \sigma^q \gamma_r$, for some integers q and r. This is Dorey's fusing rule for the fusing of particles in the real coupling affine Toda field theories[1, 15], $j + k \to r$. It is no surprise that we get a classical fusing of solitons in this case, in the sense that if we analytically continue the relative rapidity θ of the two soliton solution, made up of species j and k solitons, to this pole in $X^{jk}(\theta)$, and subject to renormalising the constants Q which occur in the solution, we get a single soliton solution of species k. We expect that in the quantum theory the solitons will couple according to this Dorey fusing rule.

The case $\gamma_j \cdot \sigma^p \gamma_k = -2$, implies that $\gamma_j + \sigma^p \gamma_k = 0$, or equivalently that the two solitons are anti-solitons of each other. This pole in $X^{jk}(\theta)$ corresponds to the breather, and actually lies at $\theta = i\pi$.

It should be noted that the zeroes of the interaction function, $X^{jk}(\theta)$, are also important and lead to additional excited modes of oscillation around the standard single solitons[17].

3 Time delays in affine Toda field theories and their S-matrix like properties

The two soliton solution in the A_{n-1} theories, of species j and k, is [20]

$$e^{-\beta\lambda_i\cdot u} = \frac{1 + Q_j\omega^{ij}W_j + Q_k\omega^{ik}W_k + X^{jk}(\theta)Q_jQ_k\omega^{i(j+k)}W_jW_k}{1 + Q_jW_j + Q_kW_k + X^{jk}(\theta)Q_jQ_kW_jW_k}, \tag{3.1}$$

with $W_r = e^{2\mu\sin(\frac{\pi r}{n})(e^{-\theta_r}x_+ - e^{\theta_r}x_-)}$, and $X^{jk}(\theta)$ is defined by equation (2.3), with $\theta = \theta_j - \theta_k$. Following the argument in [9], suppose that the velocity of the r^{th} soliton is v_r, and that $v_j > v_k$. We track the j^{th} soliton by looking at a neighbourhood in space-time of $x \sim v_jt$. In this neighbourhood W_j is of order one, and for $\kappa > 0$,

$$\begin{aligned} W_k \sim e^{\kappa(v_j - v_k)t} \quad &\rightarrow \quad \infty, \quad \text{as } t \rightarrow \infty \\ &\rightarrow \quad 0, \quad \text{as } t \rightarrow -\infty, \end{aligned} \tag{3.2}$$

so in the limit $t \rightarrow \infty$, the two soliton solution becomes

$$\begin{aligned} e^{-\beta\lambda_i.u} &= \frac{Q_k\omega^{ik}W_k + X^{jk}(\theta)Q_jQ_k\omega^{i(j+k)}W_jW_k}{Q_kW_k + X^{jk}(\theta)Q_jQ_kW_jW_k} \\ &= \omega^{ik}\left(\frac{1 + Q_jX^{jk}(\theta)\omega^{ij}W_j}{1 + Q_jX^{jk}(\theta)W_j}\right), \end{aligned} \tag{3.3}$$

and in the limit $t \rightarrow -\infty$,

$$e^{-\beta\lambda_i.u} = \frac{1 + Q_j\omega^{ij}W_j}{1 + Q_jW_j}.$$

We see that the overall affect of soliton k on the progress of soliton j through the interaction is $Q_j \rightarrow Q_jX^{jk}(\theta)$. Since the position of the soliton is proportional to $\log|Q_j|$, the time delay or phase shift is proportional to $\log X^{jk}(\theta)$.

This result is particularly simple for the A_{n-1} theories, where the single solitons have only one power of W in the numerator and denominator of $e^{-\beta\lambda_i.u}$. However in the remaining simply-laced theories there are higher powers of W, and the coefficients of the intermediate powers are not determined explicitly in the abstract formalism of Olive, Turok, and Underwood [19, 20]. It is therefore fortunate that these intermediate powers do not affect the asymptotic behaviour in time of the solutions (where we have seen that $W \rightarrow 0$, or $W \rightarrow \infty$), and that the time delay is determined completely be the ordering of the vertex operators, giving rise to $X^{jk}(\theta)$. In [9], it is demonstrated that the time delay $\Delta(\theta)$, for all simply-laced theories, is still (for Re $(\theta) > 0$)

$$\Delta(\theta) = -\frac{2h}{|\beta|^2}\log X^{jk}(\theta).$$

In [9], it is also shown that $X^{jk}(\theta)$ is real for θ real, and $0 < X^{jk}(\theta) < 1$, so that the force between distinguishable solitons must be attractive. In the following, we shall need the classical 'unitarity' condition [9]

$$X^{jk}(\theta) = X^{jk}(-\theta). \tag{3.4}$$

Also in [9], a crossing property of $X^{jk}(\theta)$ is discussed (crossing has no meaning classically) in anticipation of its relevance to the crossing property of the S-matrix, $S^{jk}(i\pi - \theta) = S^{\bar{j}k}(\theta)$. Note that a smooth analytic continuation $\theta \to i\pi - \theta$ is implicit in this crossing. We reproduce the argument here, since a modified form of the argument will be needed for the crossing of the exact S-matrix, and it will be useful to refer back.

Recall that the anti-soliton species $\bar{j}$ to the species j is defined by $\gamma_j + \sigma^p \gamma_{\bar{j}} = 0$, in fact the integer p which achieves this is known to be $p = -\frac{h}{2} - \frac{c(j)-c(\bar{j})}{4}$, see [15]. Then $X^{jk}(\theta)$ satisfies the following crossing property:

$$X^{jk}(\theta + i\pi) = X^{\bar{j}k}(\theta)^{-1}. \tag{3.5}$$

The proof is:

$$\begin{aligned}
X^{jk}(\theta + i\pi) &= \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{i\pi}{h}(2p+h+\frac{c(j)-c(k)}{2})} \right)^{-\gamma_{\bar{j}} \cdot \sigma^{p+\frac{h}{2}+\frac{c(j)-c(\bar{j})}{4}} \gamma_k} \\
&= \prod_{p'=1}^{h} \left(e^{-\theta} - e^{\frac{i\pi}{h}(2p'+\frac{(c(\bar{j})-c(k))}{2})} \right)^{-\gamma_{\bar{j}} \cdot \sigma^{p'} \gamma_k} \\
&= X^{\bar{j}k}(\theta)^{-1}.
\end{aligned}$$

Note that we do not have the expected $X^{jk}(i\pi - \theta) = X^{\bar{j}k}(\theta)$. This is because, in checking the crossing property of the semi-classical limit (1.1), we must take the crossing condition $\theta \to i\pi + \theta$, rather than $\theta \to i\pi - \theta$, so that we do not analytically continue through the imaginary θ axis [21]. During the limit, poles accumulate on the imaginary axis, which becomes a natural boundary when the limit is taken. There are two different expressions for the semi-classical limit (1.1), for Re $(\theta) > 0$ and Re $(\theta) < 0$, which are not analytic continuations of each other[1]. The best that we can do is to check the crossing property by staying within Re $(\theta) > 0$, say, by the analytic continuation $\theta \to i\pi + \theta$, and then relate the point $i\pi + \theta$ to $i\pi - \theta$ by the Hermitian analyticity property $S(i\pi + \theta)^\dagger = S(i\pi - \theta)$, for θ real [22]. The complex conjugation reverses the sign of $\log X^{\bar{j}k}(\theta)$.

Now $X^{jk}(\theta)$ has other properties which are remeniscent of the S-matrix. The poles of $X^{jk}(\theta)$ have already been discussed in the context of the classical fusing of solitons, and breathers. However it is also clear that the simple poles (due to fusing) are in precisely the same positions on the physical strip as the pole due to the fusing of particles in the exact particle Toda S-matrix $S^{jk}(\theta)$ [11, 12]. It should be noted that the fusing angles U^r_{jk}, U^j_{kr}, and U^k_{rj}, for the fusing $j + k \to \bar{r}$, are the same for particles in the particle S-matrices, and for ground state solitons in the ground state soliton S-matrices. This is so, provided the classical soliton masses all receive the same quantum corrections in the quantum theory, that is, they are all rescaled by the same constant. For the simply-laced theories, this is believed

[1] I would like to thank David Olive for discussions on this point.

to be true, see [23] for the A_n case, and the same methods as in [23] are used for the other simply-laced cases in [24]. However, for the non-simply-laced cases, which are not discussed in this paper, it is believed that the masses do not all receive the same correction [24].

If the pole in $X^{jk}(\theta)$ is simple and due to the fusing $j+k \to \bar{r}$, using the notation of section 4 for the fusing angles, and referring forward to that section, the Toda particle masses M_j obey the equation, from [15],

$$q(1) \cdot \sigma^p(\gamma_j) = iM_j e^{-\frac{i\pi}{h}(2p+\frac{(1-c(j))}{2})},$$

here $\sigma q(1) = \omega q(1)$. Hence from the fusing rule $\gamma_j + \sigma^p \gamma_k = \sigma^q \gamma_r$, we have

$$M_j + M_k e^{iU^r_{jk}} = M_{\bar{r}} e^{i\bar{U}^k_{jr}}, \tag{3.6}$$

where

$$U^r_{jk} = -\frac{\pi}{h}\left(2p + \frac{c(j)-c(k)}{2}\right),$$

and

$$\bar{U}^k_{jr} = -\frac{\pi}{h}\left(2q + \frac{c(j)-c(r)}{2}\right).$$

There are two poles in the variable e^θ in $X^{ij}(\theta)$ due to the fusing $j+k \to r$, since there are precisely two inequivalent values of p and q, p' and q', such that

$$\gamma_j + \sigma^p \gamma_k = \sigma^q \gamma_r, \quad 1 \le p \le h,$$

$$\gamma_j + \sigma^{p'} \gamma_k = \sigma^{q'} \gamma_r, \quad 1 \le p' \le h,$$

with p, p' and q, q' related by [15] $p' = h - p + \frac{c(k)-c(j)}{2}$, and $q' = h - q + \frac{c(k)-c(j)}{2}$. Now the poles in $X^{jk}(\theta)$ lie at $e^{-\theta} = e^{\frac{\pi i}{h}(2p+\frac{c(j)-c(k)}{2})}$, or

$$\theta = -\frac{i\pi}{h}(2p + \frac{c(j)-c(k)}{2}) + 2\pi n i, \tag{3.7}$$

where n is an integer, and also at the position obtained by replacing p with p'. From equation (3.6), the pole on the physical strip of the S-matrix lies at $\theta = iU^r_{jk} + 2\pi m i$, where m is the integer which allows us to take $0 < \mathrm{Im}(\theta) \le \pi$. Since $1 \le p \le h$, we must have $m = 1$, and hence the pole lies at

$$\theta = \frac{i\pi}{h}(2(h-p) + \frac{c(k)-c(j)}{2}). \tag{3.8}$$

In order to have $\mathrm{Im}(\theta) \le \pi$, we must also have $p \ge \frac{h}{2} + \frac{c(k)-c(j)}{4}$, and this is only possible for precisely one of the two allowed values of p in the fusing rule, since suppose that $p < \frac{h}{2} + \frac{c(k)-c(j)}{4}$, then for the alternative value p', where $p' = h - p + \frac{c(k)-c(j)}{2}$, and $p' > \frac{h}{2} + \frac{c(k)-c(j)}{4}$, we can choose this possibility for p, and the pole in the S-matrix on the physical strip lies at

$$\theta = iU^r_{jk} + 2\pi i = \frac{i\pi}{h}\left(2(h-p) + \frac{c(k)-c(j)}{2}\right). \tag{3.9}$$

From (3.7), we see that $X^{jk}(\theta)$ also has a pole in this position, and we have established the result.

$X^{jk}(\theta)$ also satisfies a classical bootstrap equation. Define the fusing angles $\bar{U}_{ab}^{c}$, as discussed in section 4, and suppose that the fusing $j+r\to k$ is allowed, then we have the classical bootstrap equation

$$X^{ik}(\theta)=X^{ij}(\theta-i\bar{U}_{j\bar{k}}^{r})X^{ir}(\theta+i\bar{U}_{r\bar{k}}^{j}). \tag{3.10}$$

This is proved as follows:
The fusing angles $\bar{U}_{j\bar{k}}^{r}$ and $\bar{U}_{r\bar{k}}^{j}$ can be defined through the equation

$$M_k=M_je^{-i\bar{U}_{j\bar{k}}^{r}}+M_re^{i\bar{U}_{r\bar{k}}^{j}}.$$

Again from [15], we have from $\sigma^q\gamma_k=\gamma_j+\sigma^s\gamma_r$, where s is the choice out of the two inequivalent possibilities that places the pole due to the fusing at $\theta=iU_{jr}^{\bar{k}}$ on the physical strip, that

$$\bar{U}_{j\bar{k}}^{r}=\frac{\pi}{h}\left(-2q+\frac{c(k)-c(j)}{2}\right) \tag{3.11}$$

and

$$\bar{U}_{r\bar{k}}^{j}=-\frac{\pi}{h}\left(2(s-q)+\frac{c(k)-c(r)}{2}\right). \tag{3.12}$$

We compute

$$X^{ij}(\theta-i\bar{U}_{j\bar{k}}^{r})X^{ir}(\theta+i\bar{U}_{r\bar{k}}^{j})$$

$$=\prod_{p,p'=1}^{h}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2p+\frac{c(i)-c(j)}{2})-\frac{\pi i}{h}(-2q+\frac{c(k)-c(j)}{2})}\right)^{\gamma_i\cdot\sigma^p\gamma_j}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2p'+\frac{c(i)-c(r)}{2})-\frac{\pi i}{h}(2(s-q)+\frac{c(k)-c(r)}{2})}\right)^{\gamma_i\cdot\sigma^{p'}\gamma_r}$$

if $u=p+q=p'-s+q$, then

$$=\prod_{p=1}^{h}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2u+\frac{c(i)-c(k)}{2})}\right)^{\gamma_i\cdot(\sigma^p\gamma_j+\sigma^{p'}\gamma_r)},$$

but $\sigma^p\gamma_j+\sigma^{p'}\gamma_r=\sigma^{u-q}(\gamma_j+\sigma^s\gamma_r)=\sigma^u\gamma_k$, so this expression equals

$$\prod_{u=1}^{h}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2u+\frac{c(i)-c(k)}{2})}\right)^{\gamma_i\cdot\sigma^u\gamma_k}=X^{ik}(\theta),$$

as required.

4 Axiomatics of S-matrix theory: crossing, unitarity and the bootstrap

Consider an S-matrix $S^{ab}(\theta):V_a\otimes V_b\to V_b\otimes V_a$, where θ is the relative rapidity of the two incoming solitons, and V_a is a vector spaces of charges, associated with a soliton of species a. Typically in what follows, V_a will be the a'th fundamental representation of a Lie algebra.

It must satisfy the following properties:

1. Crossing

$$S^{\bar{a}b}(\theta) = (1 \otimes C_a).[\sigma \cdot S^{ba}(i\pi - \theta)]^{t_2} \cdot \sigma \cdot (C_{\bar{a}} \otimes 1), \tag{4.1}$$

where σ is the twist map $\sigma(x \otimes y) = (y \otimes x)$, t_2 means transpose in the second space. C_a is the charge conjugation matrix from V_a to $V_{\bar{a}}$.
2. Unitarity

$$S^{ab}(\theta)S^{ba}(-\theta) = 1. \tag{4.2}$$

3. Bootstrap

Suppose that there is a pole at $\theta = iU^c_{ab}$ due to the fusing $a + b \to \bar{c}$ in $S^{ab}(\theta)$, in the direct channel, then we must have

$$m^2_{\bar{c}} = m^2_c = m^2_a + m^2_b + 2m_a m_b \cos(U^c_{ab}). \tag{4.3}$$

It also follows, by crossing, that we are allowed the fusings $bc \to \bar{a}$, and $ca \to \bar{b}$, and hence

$$U^c_{ab} + U^a_{bc} + U^b_{ca} = 2\pi.$$

It is also helpful if rewrite the mass equation (4.3) as

$$m_c = m_a e^{-i\bar{U}^b_{ac}} + m_b e^{i\bar{U}^a_{bc}}, \tag{4.4}$$

where $\bar{U}^b_{ac} = \pi - U^b_{ac}$, etc. Note that $U^c_{ab} = \bar{U}^b_{ac} + \bar{U}^a_{bc}$. This implies that we can rewrite (4.4) as

$$m_c e^{i\bar{U}^b_{ac}} = m_a + m_b e^{iU^c_{ab}} \tag{4.5}$$

Now, associated with a fusing $ab \to \bar{c}$, there is a 'fusing' of the spaces

$$V_a \otimes V_b = V_{\bar{c}} \oplus \cdots$$

Let $P_{\bar{c}}$ denote the projection from $V_a \otimes V_b$ to $V_{\bar{c}}$. The residue of $S^{ab}(\theta)$ must be proportional to $P_{\bar{c}}$, and for any third soliton d, we must have the bootstrap equation:

$$S^{d\bar{c}}(\theta) = (P_{\bar{c}} \otimes 1)(1 \otimes S^{da}(\theta + i\bar{U}^b_{ac}))(S^{db}(\theta - i\bar{U}^a_{bc}) \otimes 1)(1 \otimes P_{\bar{c}}). \tag{4.6}$$

5 The quantum Yang-Baxter equation

The quantum Yang-Baxter equation with spectral parameter is the following equation for the R-matrix

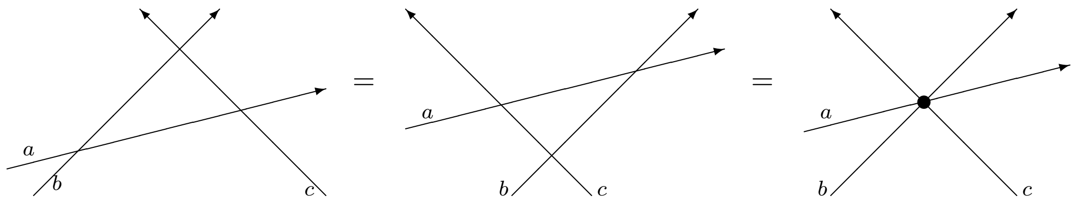
$$R^{ab}(x) : V_a \otimes V_b \to V_b \otimes V_a :$$

$$(R^{bc}(\frac{x_b}{x_c}) \otimes 1)(1 \otimes R^{ac}(\frac{x_a}{x_c}))(R^{ab}(\frac{x_a}{x_b}) \otimes 1) = (1 \otimes R^{ab}(\frac{x_a}{x_b}))(R^{ac}(\frac{x_a}{x_c}) \otimes 1)(1 \otimes R^{bc}(\frac{x_b}{x_c})), \tag{5.1}$$

both sides being a linear map

$$V_a \otimes V_b \otimes V_c \to V_c \otimes V_b \otimes V_a.$$

This is represented diagrammatically as



The affine quantum groups can provide some solutions for $R^{ab}: V_a \otimes V_b \to V_b \otimes V_a$, for certain specific representations V_a and V_b of the quantum group.

Suppose that $V_a \otimes V_b = \oplus \sum_{i=1}^{n} V_i'$, with each V_i' appearing only once. The co-product of the affine quantum group defines this decomposition, and tells us how to construct the q-dependent projection matrices P_i, from $V_a \otimes V_b$ onto V_i'. Also $\sum_{i=1}^{n} P_i = 1$, and $P_n P_m = \delta_{nm} P_n$. The R-matrix can be written

$$R^{ab}(x) = \sum_{i=1}^{n} \rho_i(x) P_i,$$

and the scalar functions $\rho_i(x)$ actually solved, since an equivalent definition of the R-matrix is that the co-product commutes with $R^{ab}(x)$. This is the celebrated work of Jimbo [25].

For $g = su(2)$, or sine-Gordon, it is possible to show, for example in the spin half representation V_1, and for the so-called homogeneous gradation, where the spectral parameter is associated only with the zero'th root of the affine algebra, that

$$R^{11}(x) =((x^{1/2}q^1 - x^{-1/2}q^{-1})P_3 + (x^{-1/2}q^1 - x^{1/2}q^{-1})P_0). \tag{5.2}$$

Here, P_3 projects onto the spin $\frac{3}{2}$ representation, and P_0 the trivial representation. We adopt the notation for writing $V \otimes W$ in vector form, as, for v_i the i^{th} entry in the vector in V, and similarly for w_i, we write $v_i \otimes w_j \in V \otimes W$ as the $(2*(i-1)+j)^{\text{th}}$ entry in the vector for $V \otimes W$.

For q the standard deformation parameter in the quantum group $U_q(su(2))$, it is easy to show from the co-product that

$$P_0 = \frac{1}{q+q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = 1 - P_0 = \frac{1}{q+q^{-1}} \begin{pmatrix} q+q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & q+q^{-1} \end{pmatrix}.$$

Hence combining P_0 and P_3 into (5.2), we have in the homogeneous gradation

$$R(x) = \begin{pmatrix} x^{1/2}q - x^{-1/2}q^{-1} & 0 & 0 & 0 \\ 0 & x^{1/2}(q-q^{-1}) & x^{1/2} - x^{-1/2} & 0 \\ 0 & x^{1/2} - x^{-1/2} & x^{-1/2}(q-q^{-1}) & 0 \\ 0 & 0 & 0 & x^{1/2}q - x^{-1/2}q^{-1} \end{pmatrix}. \tag{5.3}$$

For the soliton S-matrices, we prefer to use the principal gradation. For any simply-laced algebra g, we can move between the homogeneous and principal gradations quite easily, by setting

$$\sigma_{12} = x^{T_3/h} \otimes 1, \qquad \sigma_{21} = 1 \otimes x^{T_3/h},$$

where T_3 is the Cartan-subalgebra element of the maximal embedding of an $su(2)$ subalgebra in g. Then from [26, 27],

$$R_P(x) = \sigma_{21} R_H(x^h) \sigma_{12}^{-1}.$$

Doing this for sine-Gordon gives

$$R_P^{11}(x) = \begin{pmatrix} xq - x^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & x - x^{-1} & 0 \\ 0 & x - x^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & xq - x^{-1}q^{-1} \end{pmatrix}.$$

Now it is possible to show, for example there is a proof in [4], that $R^{ab}(x)$ satisfies the crossing condition (4.1), for V_a and V_b the fundamental representations, in either the principal or homogeneous gradations. This can be checked explicitly for sine-Gordon by the reader.

It is also possible to show, for A_n, that $R^{ab}(x)$ satisfies the bootstrap equation (4.6), for V_a and V_b again the fundamental representations. This is usually called 'fusion' in the literature. However for the D and E series of algebras, it is only possible to show this for R intertwining between tensor products of larger spaces $W_a \supset V_a$, but with $W_1 = V_1$ [28]. This is an argument for saying that R-matrices do not exist for R intertwining between the smaller spaces $V_a \otimes V_b$ (if either a or b is different from 1), which is vindicated by the method for calculating R-matrices using projections, see section 11.

We also note that for $R^{11}(x)$ defined by (5.2) in sine-Gordon, for the homogeneous gradation

$$\begin{aligned} R_H^{11}(x) R_H^{11}(x^{-1}) &= (x^{1/2}q - x^{-1/2}q^{-1})(x^{-1/2}q - x^{1/2}q^{-1})(P_3 + P_0) \\ &= \frac{(1 - xq^2)(1 - x^{-1}q^2)}{q^2}.1. \end{aligned}$$

The only change for the principal gradation is $x \to x^2$, and

$$R_P^{11}(x) R_P^{11}(x^{-1}) = \frac{(1 - x^2q^2)(1 - x^{-2}q^2)}{q^2}.1. \tag{5.4}$$

6 The 'regularised' quantum dilogarithm

The quantum dilogarithm [29, 7] is defined as

$$S_q(w) = \prod_{k=0}^{\infty} (1 + q^{2n+1}w) = \exp\Big(\sum_{k=1}^{\infty} \frac{(-w)^k}{k(q^k - q^{-k})} \Big). \tag{6.1}$$

It satisfies the key defining property

$$\frac{S_q(qw)}{S_q(q^{-1}w)} = \frac{1}{1+w}. \tag{6.2}$$

The first expression converges for $|q| < 1$, and the second for $|q| \neq 1$, and $|w| < 1$. The classical dilogarithm, for $|w| < 1$, is defined as

$$L_2(w) = -\int_0^w \frac{dz}{z} \log(1-z) = \sum_{k=1}^{\infty} \frac{w^k}{k^2}, \tag{6.3}$$

so the second equation of (6.1) is a sort of q-deformed exponential of the classical dilogarithm (6.3). Furthermore in the limit $\epsilon \to 0$, $\epsilon < 0$ and $q = e^{\epsilon}$, it is easy to see that, to leading order

$$S_q(w) \sim e^{\frac{1}{2\epsilon} L_2(-w)},$$

and this further justifies the term 'quantum dilogarithm' for (6.1).

Unfortunately (6.1) suffers from a serious defect which means that as it stands it cannot play a rôle in the integrable models discussed in this paper, simply because we expect that $|q| = 1$. The second expression in (6.1) is seen to diverge, since if q is a root of unity then a term in the series is infinite, otherwise q^{2k} becomes arbitrarily close to 1 an infinite number of times, and so the series must be greater than any given bound. Nevertheless, the situation can be repaired because the function, introduced in [7],

$$\hat{S}_q(w) = \exp\left(\frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{x} \frac{(w)^{-ix}}{\sinh(\pi x)\sinh(\mu x)}\right), \tag{6.4}$$

where $q = e^{i\mu}$, and the contour goes above the pole at the origin, satisfies

$$\frac{\hat{S}_q(qw)}{\hat{S}_q(q^{-1}w)} = \frac{1}{1+w}. \tag{6.5}$$

The same property as (6.2). The integral (6.4) converges if $\mathrm{Im}(\log(w)) < \pi + \mu$, we then use the above functional equation to define it for all $\log(w)$. To show this property (6.5), we compute

$$\frac{\hat{S}_q(qw)}{\hat{S}_q(q^{-1}w)} = \exp\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x} \frac{(w)^{-ix}}{\sinh(\pi x)}\right),$$

closing the contour in the upper-half plane if $|w| < 1$, the lower-half plane if $|w| > 1$, and summing up the residue contributions from simple poles on the imaginary axis, taking care of a residue contribution from the origin when closing the contour in the lower-half plane, as shown in [7].

We take (6.4) as a suitable definition of a 'regularised' quantum dilogarithm which should replace (6.1), when $|q| = 1$. To further justify the term 'dilogarithm', we also show that the classical dilogarithm is obtained in the limit $\mu \to 0$, this will also be important for the calculations to follow.

The leading order behaviour in this limit is given by (for $|z| < 1$), closing the contour in the upper half plane,

$$\begin{aligned} \log \hat{S}_q(z) &= \frac{1}{4\mu}\int_{-\infty}^{\infty} \frac{dx}{x^2} \frac{z^{-ix}}{\sinh \pi x} \\ &= \frac{2\pi i}{4\mu} \sum_{n=1}^{\infty} \frac{z^n}{-n^2\pi(-1)^n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-i}{2\mu} L_2(-z) \\
&= \frac{i}{2\mu} \int_0^z \frac{dw}{w} \log(1+w).
\end{aligned} \tag{6.6}
$$

For $|z| > 1$, closing the contour in the lower half plane,

$$
\begin{aligned}
\log(\hat{S}_q(z)) &= \frac{1}{4\mu} \int_{-\infty}^{\infty} \frac{dx}{x^2} \frac{z^{-ix}}{\sinh \pi x} \\
&= -\frac{2\pi i}{4\mu} \sum_{n=-1}^{-\infty} \frac{z^n}{-n^2 \pi (-1)^n} - 2\pi i \mathrm{res}(0) \\
&= \frac{i}{2\mu} \sum_{n=1}^{\infty} \frac{(-z)^{-n}}{n^2} + \frac{i}{4\mu} (\log z)^2 \\
&= \frac{i}{2\mu} L_2(-z^{-1}) + \frac{i}{4\mu} (\log z)^2 \\
&= -\frac{i}{2\mu} \int_0^{z^{-1}} \frac{dw}{w} \log(1+w) + \frac{i}{4\mu} (\log z)^2.
\end{aligned} \tag{6.7}
$$

Now observe that if we change the variable of integration of (6.4) by $x \to (\pi/\mu)x$, then

$$
\hat{S}_q(w) = \exp\left(\frac{1}{4} \int_{-\infty}^{\infty} \frac{dx}{x} \frac{(w)^{-i\frac{\pi}{\mu}x}}{\sinh(\frac{\pi^2}{\mu}x)\sinh(\pi x)} \right) = \hat{S}_{\tilde{q}}(w^{\pi/\mu}), \tag{6.8}
$$

where $\tilde{q} = e^{\frac{\pi^2}{\mu} i}$.

We can also expand the function, in order to study its poles and zeroes. Setting $e^{ip} = w$, we have

$$
\hat{S}_\mu(w) = e^{-\frac{ip^2}{8\mu} - \frac{i}{24}(\mu + \frac{\pi^2}{\mu})} \prod_{k,l=0}^{\infty} \left(\frac{p + \pi(2k+1) + \mu(2l+1)}{-p + \pi(2k+1) + \mu(2l+1)} \right). \tag{6.9}
$$

This can be written as an infinite product of gamma functions, using the Weierstrass product formula for the gamma function. We can do this in two different ways, by taking the product over k or l. Thus

$$
\begin{aligned}
\hat{S}_\mu(w) &= e^{-\frac{ip^2}{8\mu} - \frac{i}{24}(\mu + \frac{\pi^2}{\mu})} \prod_{l=0}^{\infty} \left(\frac{\Gamma(-\frac{p}{2\pi} - \frac{1}{2\pi} + \frac{\mu}{2\pi}(2l+1))}{\Gamma(\frac{p}{2\pi} - \frac{1}{2\pi} + \frac{\mu}{2\pi}(2l+1))} \right), \\
\hat{S}_\mu(w) &= e^{-\frac{ip^2}{8\mu} - \frac{i}{24}(\mu + \frac{\pi^2}{\mu})} \prod_{k=0}^{\infty} \left(\frac{\Gamma(-\frac{p}{2\mu} - \frac{1}{2\mu} + \frac{\pi}{2\mu}(2k+1))}{\Gamma(\frac{p}{2\mu} - \frac{1}{2\mu} + \frac{\pi}{2\mu}(2k+1))} \right).
\end{aligned} \tag{6.10}
$$

The duality property (6.8) under $\mu \to \pi^2/\mu$ can be rederived from (6.9) by swapping k and l in the product and multiplying the numerator and denominator by π/μ.

We also note the property,

$$
\hat{S}_\mu(w)\hat{S}_\mu(w^{-1}) = e^{\frac{i(\log(w))^2}{4\mu} - \frac{i}{12}(\mu + \frac{\pi^2}{\mu})}, \tag{6.11}
$$

derived from computing the integral $\hat{S}_\mu(w)\hat{S}_\mu(w^{-1})$, the contour becoming a small circle around the orgin, so only the residue at the origin contributes.

In what follows we replace the notation $\hat{S}_q(z)$ with $S_q(z)$.

7 The sine-Gordon solution (Zamolodchikov-Zamolodchikov)

For $g = su(2)$, we recall the equation for $R^{11}(\theta)$, as given in section 5,

$$S^{11}(\theta) = v(x)\begin{pmatrix} xq - x^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & q-q^{-1} & x-x^{-1} & 0 \\ 0 & x-x^{-1} & q-q^{-1} & 0 \\ 0 & 0 & 0 & xq-x^{-1}q^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} S(\theta) & 0 & 0 & 0 \\ 0 & S_R(\theta) & S_T(\theta) & 0 \\ 0 & S_T(\theta) & S_R(\theta) & 0 \\ 0 & 0 & 0 & S(\theta) \end{pmatrix}.$$

Here, following from the standard notation in [5], $S(\theta)$ is the identical soliton-soliton process (or anti-soliton–anti-soliton), $S_R(\theta)$ the soliton–anti-soliton reflection process, and $S_T(\theta)$ the soliton–anti-soliton transmission process. From the crossing property of $R^{11}(x)$ and (5.4) in section 5, the crossing and unitarity equations for $v(x)$ are respectively

$$v(x) = v(x^{-1}q^{-1}), \tag{7.1}$$

$$v(x^{-1})v(x) = \frac{q^2}{(1-x^2q^2)(1-x^{-2}q^2)}. \tag{7.2}$$

Here,

$$x = e^{\frac{8\pi\theta}{\gamma}}, \quad q = e^{-\frac{8\pi^2 i}{\gamma}} = -e^{-\frac{8\pi^2 i}{|\beta|^2}}, \quad \gamma = \frac{|\beta|^2}{1-\frac{|\beta|^2}{8\pi}},$$

and we see that the crossing condition $\theta \to i\pi - \theta$ implies $x \to x^{-1}q^{-1}$.
Now an infinite family of solutions to these equations, labelled by the integer n, is given by:

$$v(x) = \frac{q}{(1-x^2q^2)}\frac{X_q(x)}{X_q(x^{-1})} \tag{7.3}$$

where

$$X_q(x) = \frac{S_{q^{-2}}(e^{i\pi+2\pi ni}x^2q^2)}{S_{q^{-2}}(e^{i\pi+2\pi ni}x^2)} \tag{7.4}$$

Proof
We check crossing (7.1)

$$X_q(x^{-1}q^{-1}) = \frac{S_{q^{-2}}(e^{i\pi+2\pi ni}x^{-2})}{S_{q^{-2}}(e^{i\pi+2\pi ni}x^{-2}q^{-2})} = X_q(x^{-1})^{-1}(1-x^{-2}),$$

in the last step we have used the property (6.5). Similarly

$$X_q(xq) = \frac{S_{q^{-2}}(e^{i\pi+2\pi ni}x^2q^4)}{S_{q^{-2}}(e^{i\pi+2\pi ni}x^2q^2)} = X_q(x)^{-1}(1-x^2q^2).$$

Then

$$v(x^{-1}q^{-1}) = \frac{q}{1-x^{-2}}\frac{X_q(x^{-1}q^{-1})}{X_q(xq)} = \frac{q}{1-x^2q^2}\frac{X_q(x)}{X_q(x^{-1})} = v(x),$$

as required.

Observe that the unitarity equation (7.2) follows automatically.

Comparison with the Zamolodchikov-Zamolodchikov solution

We will see that the above solution for $n = 0$ is the same as the Zamolodchikov-Zamolodchikov solution [5].

Proof

We work with

$$S_T(\theta) = \frac{q(x-x^{-1})}{1-x^2q^2}v(x) = qx\frac{S_{q^{-2}}(e^{i\pi}x^2q^2)S_{q^{-2}}(e^{i\pi}x^{-2})}{S_{q^{-2}}(e^{i\pi}x^{-2}q^{-2})S_{q^{-2}}(e^{i\pi}x^2q^4)}.$$

Here the common intersection of ranges of θ for which each integral in each of the four quantum dilogarithms separately converges is

$$\pi > \mathrm{Im}\,(\theta) > \pi - \frac{\gamma}{8}.$$

Then

$$S_T(\theta) = qx\exp\Big(\frac{1}{4}\int_{-\infty}^{\infty}\frac{dy}{y}\frac{(e^{i\pi}x^2q^2)^{-iy} + (e^{i\pi}x^{-2})^{-iy} - (e^{i\pi}x^{-2}q^{-2})^{-iy} - (e^{i\pi}x^2q^4)^{-iy}}{\sinh(\pi y)\sinh(\mu y)}\Big)$$

$$\text{for} \quad q^{-2} = e^{i\mu}, \qquad \mu = \frac{16\pi^2}{\gamma},$$

and

$$\begin{aligned}(e^{i\pi}x^2q^2)^{-iy} + \quad (e^{i\pi}x^{-2})^{-iy} \quad &-(e^{i\pi}x^{-2}q^{-2})^{-iy} - (e^{i\pi}x^2q^4)^{-iy} \\ &= \quad (e^{-16\pi i\theta y/\gamma - \mu y} - e^{16\pi i\theta y/\gamma + \mu y})(1-e^{-\mu y})e^{\pi y} \\ &= \quad 4\sinh(-16\pi i\theta y/\gamma - \mu y)\sinh(\frac{\mu y}{2})e^{\pi y - \frac{\mu y}{2}}. \end{aligned} \tag{7.5}$$

Let

$$\begin{aligned} J = \log(S_T(\theta)) - \log(qx) \quad &= \quad \frac{1}{4}\int_{-\infty}^{\infty}\frac{dy}{y}\frac{(e^{i\pi}x^2q^2)^{-iy} + (e^{i\pi}x^{-2})^{-iy} - (e^{i\pi}x^{-2}q^{-2})^{-iy} - (e^{i\pi}x^2q^4)^{-iy}}{\sinh(\pi y)\sinh(\mu y)} \\ &= \quad -\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sinh(16\pi i\theta y/\gamma + \mu y)e^{(\pi-\frac{\mu}{2})y}}{2\cosh(\frac{\mu y}{2})\sinh(\pi y)}, \end{aligned} \tag{7.6}$$

and let $y \to -y$, then the contour passes below the pole at the origin. Moving the contour through this pole gives a positive residue contribution from the origin.

$$J = \int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sinh(16\pi i\theta y/\gamma + \mu y)e^{-(\pi-\frac{\mu}{2})y}}{2\cosh(\frac{\mu y}{2})\sinh(\pi y)} + 2\pi i.\mathrm{res}(0),$$

where the contour here is the same as in (7.6).

Now

$$2\pi i.\mathrm{res}(0) = 2\pi i\frac{1}{2\pi}(\frac{16\pi i\theta}{\gamma} + \frac{16\pi^2}{\gamma}) = \frac{-16\pi\theta}{\gamma} + \frac{16\pi^2 i}{\gamma},$$

and

$$J = -\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sinh(16\pi i\theta y/\gamma+\mu y)}{2\cosh(\frac{\mu y}{2})\sinh(\pi y)}\frac{1}{2}(e^{(\pi-\mu/2)y}-(e^{-(\pi-\mu/2)y}+2\pi i.\mathrm{res}(0)))$$

$$= \frac{2\pi i.\mathrm{res}(0)}{2}-\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sinh(16\pi i\theta y/\gamma+\mu y)\sinh(\pi-\mu/2)y}{2\cosh(\frac{\mu y}{2})\sinh(\pi y)}. \quad (7.7)$$

After the change of integration variable, $y \to \frac{\gamma}{16\pi^2}y$,

$$\log(S_T(\theta)) - \log(qx) = \frac{-8\pi\theta}{\gamma}+\frac{8\pi^2 i}{\gamma}+\frac{i}{2}\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sin((i-\frac{\theta}{\pi})y)\sinh(\frac{\gamma}{16\pi}-\frac{1}{2})y}{\cosh(\frac{y}{2})\sinh(\frac{\gamma y}{16\pi})}. \quad (7.8)$$

Since $\log(qx) = \frac{8\pi\theta}{\gamma}-\frac{8\pi^2 i}{\gamma}$, we have the result

$$S_T(\theta) = \exp\Big(\frac{i}{2}\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\sin((i-\frac{\theta}{\pi})y)\sinh(\frac{\gamma}{16\pi}-\frac{1}{2})y}{\cosh(\frac{y}{2})\sinh(\frac{\gamma y}{16\pi})}\Big). \quad (7.9)$$

This is the solution due to Karowski *et al* [6], and converges in the stated range

$$\pi > \mathrm{Im}\,(\theta) > \pi - \frac{\gamma}{8}.$$

We also expand our result for $n = 0$ in terms of an infinite double product, using the expansion (6.9) for the quantum dilogarithm – this is not strictly needed to establish the identity with the Zamolodchikov-Zamolodchikov solution [5], since this is guaranteed through the Karowski integral formula above, but the result, of course, agrees with [5], after we move to the description using gamma functions.

$$\begin{aligned} S_T(\theta) &= qx.e^{\frac{8\pi^2 i}{\gamma}}e^{\frac{-8\pi\theta}{\gamma}}.\prod_{k,l=0}^{\infty}\Big(\frac{\pi(2k+2)+\mu(2l)-\frac{16\pi\theta i}{\gamma}}{\pi(2k)+\mu(2l+2)+\frac{16\pi\theta i}{\gamma}}\Big)\Big(\frac{\pi(2k+2)+\mu(2l+1)+\frac{16\pi\theta i}{\gamma}}{\pi(2k)+\mu(2l+1)-\frac{16\pi\theta i}{\gamma}}\Big) \\ &\times \Big(\frac{\pi(2k)+\mu(2l)-\frac{16\pi\theta i}{\gamma}}{\pi(2k+2)+\mu(2l+2)+\frac{16\pi\theta i}{\gamma}}\Big)\Big(\frac{\pi(2k)+\mu(2l+3)+\frac{16\pi\theta i}{\gamma}}{\pi(2k+2)+\mu(2l-1)-\frac{16\pi\theta i}{\gamma}}\Big), \end{aligned}$$

so

$$\begin{aligned} S_T(\theta) &= \prod_{k,l=0}^{\infty}\Big(\frac{k+1+\frac{8\pi}{\gamma}2l-\frac{8\theta i}{\gamma}}{k+\frac{8\pi}{\gamma}(2l+2)+\frac{8\theta i}{\gamma}}\Big)\Big(\frac{k+1+\frac{8\pi}{\gamma}(2l+1)+\frac{8\theta i}{\gamma}}{k+\frac{8\pi}{\gamma}(2l+1)-\frac{8\theta i}{\gamma}}\Big) \\ &\times \Big(\frac{k+\frac{8\pi}{\gamma}(2l)-\frac{8\theta i}{\gamma}}{k+1+\frac{8\pi}{\gamma}(2l+2)+\frac{8\theta i}{\gamma}}\Big)\Big(\frac{k+\frac{8\pi}{\gamma}(2l+3)+\frac{8\theta i}{\gamma}}{k+1+\frac{8\pi}{\gamma}(2l-1)-\frac{8\theta i}{\gamma}}\Big). \end{aligned} \quad (7.10)$$

The reflectionless result

The reflection process, given by $S_R(\theta)$, vanishes at the values of the coupling $q^2 = 1$, or $\frac{8\pi}{\gamma} = N$, where N is an integer. We relate these reflectionless conditions on the coupling to q a primitive even root of unity, in the hope that the expression (7.3) will simplify. In order to do this we must pass to the dual description of $S_T(\theta)$, using the duality property (6.8) :

$$S_T(\theta) = qx\frac{S_{\tilde q^{-2}}(e^{-i\pi}\tilde q^{-2}e^{\theta})S_{\tilde q^{-2}}(\tilde q^{-2}e^{-\theta})}{S_{\tilde q^{-2}}(e^{-2i\pi}\tilde q^{-2}e^{\theta})S_{\tilde q^{-2}}(e^{i\pi}\tilde q^{-2}e^{-\theta})} \quad (7.11)$$

At these reflectionless values of the coupling $\tilde{q}^{-2} = e^{\frac{i\gamma}{16}} = e^{\frac{i\pi}{2N}}$. Setting $2N = n$, and using the quantum dilogarithm property (6.5) for $S_{\tilde{q}^{-2}}(w)$:

$$\frac{S_{\tilde{q}^{-2}}(w)}{S_{\tilde{q}^{-2}}(we^{\frac{2\pi i}{n}})} = (1 + we^{\frac{i\pi}{n}}),$$

$$\frac{S_{\tilde{q}^{-2}}(we^{\frac{2\pi i}{n}})}{S_{\tilde{q}^{-2}}(we^{\frac{4\pi i}{n}})} = (1 + we^{\frac{3\pi i}{n}}),$$

$$\vdots$$

$$\frac{S_{\tilde{q}^{-2}}(we^{\frac{(n-2)\pi i}{n}})}{S_{\tilde{q}^{-2}}(we^{i\pi})} = (1 + we^{\frac{(n-1)\pi i}{n}})$$

so

$$\frac{S_{\tilde{q}^{-2}}(w)}{S_{\tilde{q}^{-2}}(we^{i\pi})} = (1 + we^{\frac{\pi i}{n}})(1 + we^{\frac{3\pi i}{n}}) \cdots (1 + we^{\frac{(n-1)\pi i}{n}}), \tag{7.12}$$

and

$$\frac{S_{\tilde{q}^{-2}}(we^{-i\pi})}{S_{\tilde{q}^{-2}}(w)} = (1 + we^{\frac{-\pi i}{n}})(1 + we^{\frac{-3\pi i}{n}}) \cdots (1 + we^{\frac{-(n-1)\pi i}{n}}). \tag{7.13}$$

Thus

$$\begin{aligned} S_T(\theta) &= e^{iN\pi} e^{N\theta} \frac{(1 + e^{\frac{2\pi i}{n}} e^{-\theta})(1 + e^{\frac{4\pi i}{n}} e^{-\theta}) \cdots (1 + e^{\frac{n\pi i}{n}} e^{-\theta})}{(1 + e^{\frac{2\pi i}{n}} e^{\theta})(1 + e^{\frac{4\pi i}{n}} e^{\theta}) \cdots (1 + e^{\frac{n\pi i}{n}} e^{\theta})} \\ &= e^{iN\pi} \frac{(1 + e^{\frac{-i\pi}{N}} e^{\theta})(1 + e^{\frac{-2i\pi}{N}} e^{\theta}) \cdots (1 + e^{\frac{-Ni\pi}{N}} e^{\theta})}{(e^{\frac{-i\pi}{N}} + e^{\theta})(e^{\frac{-2i\pi}{N}} + e^{\theta}) \cdots (e^{\frac{-Ni\pi}{N}} + e^{\theta})}. \end{aligned} \tag{7.14}$$

This expression was the first result for the sine-Gordon S-matrix [30], originally obtained by extrapolating from the time delay, $\Delta(\theta) = \frac{8}{|\beta|^2} \log\Big(\frac{1-e^{\theta}}{1+e^{\theta}}\Big)^2$, by using the formula

$$i \int_0^{\theta} d\theta' \log\left(\frac{e^{\theta'} - 1}{e^{\theta'} + 1}\right)^2 = \int_0^{\pi} dx \log\left(\frac{e^{\theta} e^{-ix} + 1}{e^{-ix} + e^{\theta}}\right). \tag{7.15}$$

The full S-matrix (7.10), at arbitrary values of the coupling, was later obtained by extrapolating further from this result in [5], and such an extrapolation was carried out explicitly in an elegant way in [31]. However, we shall see that the method involving quantum dilogarithms is a more efficient way of extrapolating from the time delay to the full S-matrix, because it is not clear, in general, how to find an analogue of (7.15) for the other cases.

There are an infinite family of solutions (7.3) labelled by the integer n. These can all be written as the Zamolodchikov solution (with $n = 0$), multiplied by CDD factors, by repeated application of (6.5), in the dual description.

8 The semi-classical limit for sine-Gordon

We work with the soliton-soliton process

$$S(\theta) = -(xq - x^{-1}q^{-1})v(x) = x^{-1} \frac{X_q(x)}{X_q(x^{-1})}.$$

Taking the limit as $\gamma \to 0$ of $S_{q^{-2}}(w)$ is problematical since $q^{-2} = e^{\frac{16\pi^2 i}{\gamma}}$, and so we pass to the dual description using property (6.8) of the quantum dilogarithm. This limit then corresponds to $\tilde{q} \to 1$, and it has already been shown that we recover the classical dilogarithm in this limit.

With $\tilde{q}^{-2} = e^{\frac{i\gamma}{16}}$, we have

$$\begin{aligned} X_q(x) &= \frac{S_{\tilde{q}^{-2}}((e^{i\pi+2\pi ni}x^2q^2)^{\gamma/16\pi})}{S_{\tilde{q}^{-2}}((e^{i\pi+2\pi ni}x^2)^{\gamma/16\pi})} \\ &= \frac{S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{\theta}e^{-i\pi})}{S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{\theta})}. \end{aligned} \tag{8.1}$$

Taking Re $(\theta) < 0$, the leading order behaviour of this limit is (from equation (6.6))

$$\lim_{\gamma\to 0} S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{\theta}) = \exp\left(\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log(1+w)\right),$$

and

$$\lim_{\gamma\to 0} S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{\theta}e^{-i\pi}) = \exp\left(\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log(1-w)\right),$$

and we see that

$$\lim_{\gamma\to 0} X_q(x) = \exp\left(\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log\left(\frac{1-w}{1+w}\right)\right). \tag{8.2}$$

We see how the structure of the time delay for sine-Gordon, namely $\Delta(\theta') = \frac{8}{\beta^2}\log\left(\frac{1-e^{\theta'}}{1+e^{\theta'}}\right)^2$, enters $X_q(x)$, in fact $X_q(x)$ can be thought of as a sort of q-deformed $X^{ij}(\theta)$, the normal ordering coefficient of two classical vertex operators.

Now

$$X_q(x^{-1}) = \frac{S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{-\theta}e^{-i\pi})}{S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{-\theta})},$$

and from equation (6.7),

$$\lim_{\gamma\to 0} S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{-\theta}) = \exp\left(-\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log(1+w) + \frac{4i}{\gamma}\theta^2\right),$$

$$\lim_{\gamma\to 0} S_{\tilde{q}^{-2}}(\tilde{q}^{-2-4n}e^{-\theta}e^{-i\pi}) = \exp\left(-\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log(1-w) + \frac{4i}{\gamma}(\theta+i\pi)^2\right),$$

and

$$\lim_{\gamma\to 0} X_q(x^{-1}) = \exp\left(-\frac{8i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log\left(\frac{1-w}{1+w}\right)\right).e^{\frac{4i}{\gamma}(-\pi^2+2\pi i\theta)}.$$

Thus

$$\begin{aligned} \lim_{\gamma\to 0} x^{-1}\frac{X_q(x)}{X_q(x^{-1})} &= \exp\left(\frac{16i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log\left(\frac{1-w}{1+w}\right)\right).x^{-1}e^{\frac{8\pi\theta}{\gamma}+\frac{4i\pi^2}{\gamma}} \\ &= e^{\frac{4i\pi^2}{\gamma}}\exp\left(\frac{16i}{\gamma}\int_0^{e^{\theta}}\frac{dw}{w}\log\left(\frac{1-w}{1+w}\right)\right) \end{aligned}$$

$$= \exp\Big(\frac{16i}{\gamma}\int_1^{e^\theta}\frac{dw}{w}\log\Big(\frac{1-w}{1+w}\Big)\Big), \tag{8.3}$$

since

$$\int_0^1 \frac{dw}{w}\log\Big(\frac{1-w}{1+w}\Big) = \frac{1}{2}\int_{-1}^1 \frac{dw}{w}\log\Big(\frac{1-w}{1+w}\Big) = -\frac{\pi^2}{4}.$$

So, for $\mathrm{Re}(\theta) < 0$,

$$\lim_{\gamma\to 0} S(\theta) = \exp\Big(\frac{8i}{\gamma}\int_0^\theta d\theta' \log\Big(\frac{1-e^{\theta'}}{1+e^{\theta'}}\Big)^2\Big). \tag{8.4}$$

This is the correct form of the semi-classical limit for $S(\theta)$.

9 The $su(3)$ solution

We label the fundamental representations of $su(3)$ λ_1, by 3, and λ_2 by $\bar{3}$, the trivial representation by 1, the adjoint representation by 8. From the decomposition $3\otimes 3 = 6\oplus\bar{3}$, we have, in the homogeneous gradation

$$R^{11}(x) = (x^{1/2}q - x^{-1/2}q^{-1})P_6^{33} + (x^{-1/2}q - x^{1/2}q^{-1})P_{\bar{3}}^{33}, \tag{9.1}$$

where P_6^{33} and $P_{\bar{3}}^{33}$ are the projections from $3\otimes 3\to 6$ and $3\otimes 3\to\bar{3}$ respectively. We also have

$$P_6^{33} + P_{\bar{3}}^{33} = 1, \tag{9.2}$$

and

$$(P_6^{33})^2 = P_6^{33}, \quad (P_{\bar{3}}^{33})^2 = P_{\bar{3}}^{33}, \quad P_{\bar{3}}^{33}P_6^{33} = P_6^{33}P_{\bar{3}}^{33} = 0. \tag{9.3}$$

Similarly, from the decomposition $3\otimes\bar{3} = 8\oplus 1$, we have, in the homogeneous gradation,

$$R^{12}(x_{12}) = (x_{12}^{1/2}q^{3/2} - x_{12}^{-1/2}q^{-3/2})P_8^{3\bar{3}} + (x_{12}^{-1/2}q^{3/2} - x_{12}^{1/2}q^{-3/2})P_1^{3\bar{3}}, \tag{9.4}$$

and

$$R^{21}(x_{21}) = (x_{21}^{1/2}q^{3/2} - x_{21}^{-1/2}q^{-3/2})P_8^{\bar{3}3} + (x_{21}^{-1/2}q^{3/2} - x_{21}^{1/2}q^{-3/2})P_1^{\bar{3}3}, \tag{9.5}$$

where $x_{12} = x_{21}^{-1}$. Since $3\otimes\bar{3}\neq\bar{3}\otimes 3$ we must replace equations (9.2) and (9.3) by

$$P_8^{\bar{3}3}P_8^{3\bar{3}} + P_1^{\bar{3}3}P_1^{3\bar{3}} = 1,$$

and

$$P_1^{\bar{3}3}P_8^{3\bar{3}} = P_8^{\bar{3}3}P_1^{3\bar{3}} = 0.$$

Using the Ansatz $S^{ij}(\theta) = v^{ij}(\theta)R^{ij}(x_{ij})$, for the S-matrix, and $x = x_{12} = x_{11} = e^{\frac{4\pi\theta}{\gamma}}$, for the spectral parameter x_{ij} in the principal gradation, and $q = e^{-\frac{4\pi^2}{\gamma}i}$. γ is defined as

$$\gamma = \frac{|\beta|^2}{1-\frac{|\beta|^2}{4\pi}}.$$

Now the unitarity condition is $S^{ji}(-\theta)S^{ij}(\theta) = 1$. Using the Ansatz for the S-matrix, we then have

$$v^{ji}(-\theta)v^{ij}(\theta)R^{ji}(x_{ji})R^{ij}(x_{ij}) = 1.$$

We also impose $v^{ij}(\theta) = v^{ji}(\theta)$. In this $su(3)$, case we compute

$$\begin{aligned} R^{11}(x^{-1})R^{11}(x) &= \frac{(1-xq^2)(1-x^{-1}q^2)}{q^2}, \\ R^{21}(x^{-1})R^{12}(x) &= (x^{1/2}q^{3/2} - x^{-1/2}q^{-3/2})(x^{-1/2}q^{3/2} - x^{1/2}q^{-3/2}). \end{aligned} \tag{9.6}$$

However the equations (9.1),(9.4) and (9.5) were for the homogeneous gradation, to move to the principal gradation we must conjugate by the matrix σ_{12}

$$R_P(x) = \sigma_{21} R_H(x^h) \sigma_{12}^{-1},$$

the only effect this will have on the unitarity equations (9.6) is to replace x by x^3, so that in the principal gradation these equations read:

$$\begin{aligned} R^{11}(x^{-1})R^{11}(x) &= \frac{(1-x^3q^2)(1-x^{-3}q^2)}{q^2} \\ R^{21}(x^{-1})R^{12}(x) &= (x^{3/2}q^{3/2} - x^{-3/2}q^{-3/2})(x^{-3/2}q^{3/2} - x^{3/2}q^{-3/2}). \end{aligned} \tag{9.7}$$

These equations in turn imply the following unitarity conditions for the scalar factors $v^{11}(\theta)$ and $v^{12}(\theta)$:

$$\begin{aligned} v^{11}(-\theta)v^{11}(\theta) &= \frac{q^2}{(1-x^3q^2)(1-x^{-3}q^2)}, \\ v^{12}(-\theta)v^{12}(\theta) &= \frac{1}{(x^{3/2}q^{3/2} - x^{-3/2}q^{-3/2})(x^{-3/2}q^{3/2} - x^{3/2}q^{-3/2})}. \end{aligned} \tag{9.8}$$

The following crossing condition must also be satisfied under the crossing transformation $\theta \to i\pi - \theta$,

$$v^{11}(i\pi - \theta) = v^{21}(\theta) = v^{12}(\theta), \tag{9.9}$$

this is because with the normalisations chosen for $R^{ij}(x)$, fixed by equations (9.1), (9.4) and (9.5), the matrices $R^{11}(x)$ and $R^{12}(x)$ cross precisely into each other under the crossing transformation $x \to x^{-1}q^{-1}$, and the application of the crossing matrices, with no extraneous x-dependent factors $(1 - x^h q^p)$. Bearing in mind that the classical time delays for $su(3)$ are given by (2.3):

$$\begin{aligned} X^{11}(\theta) &= \frac{(1-e^{\theta})^2}{(1-e^{\frac{2\pi i}{3}}e^{\theta})(1-e^{\frac{4\pi i}{3}}e^{\theta})}, \\ X^{12}(\theta) &= \frac{(1-e^{-\frac{\pi i}{3}}e^{\theta})(1-e^{\frac{\pi i}{3}}e^{\theta})}{(1+e^{\theta})^2}, \end{aligned} \tag{9.10}$$

we propose the following solution for $v^{11}(\theta)$,

$$v^{11}(x) = \frac{-qe^{\theta/2}}{(1-x^3q^2)} \frac{X_q^{11}(x)}{X_q^{11}(x^{-1})}, \tag{9.11}$$

where

$$X_q^{11}(x) = \frac{S_{q^{-3}}(e^{i\pi}x^3q^3)}{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q)S_{q^{-3}}(e^{-i\pi}x^3q^{-1})}}. \tag{9.12}$$

By extending the semi-classical limit calculation as $\gamma \to 0$ for sine-Gordon, section 8, we see that, for $\mathrm{Re}(\theta) < 0$, (also from (6.6)), after passing to the dual description using (6.8),

$$X_q^{11}(\theta) = \frac{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{-i\pi})}{\sqrt{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{-\frac{i\pi}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{\frac{i\pi}{3}})}},$$

in the limit $\gamma \to 0$,

$$\lim_{\gamma\to 0} X_q^{11}(\theta) \sim e^{\frac{6i}{\gamma}\int_{-\infty}^{\theta} d\theta' \log(X^{11}(\theta')^{1/2})},$$

and for $\mathrm{Re}(\theta) < 0$, from (6.7)

$$\lim_{\gamma\to 0} X_q^{11}(-\theta) \sim e^{-\frac{6i}{\gamma}\int_{-\infty}^{\theta} d\theta' \log(X^{11}(-\theta')^{1/2})}.e^{\frac{6i}{2\gamma}((-\theta-i\pi)^2-\frac{1}{2}(-\theta-\frac{i\pi}{3})^2-\frac{1}{2}(-\theta+\frac{i\pi}{3})^2)}.$$

We use the classical property $X^{ij}(\theta) = X^{ij}(-\theta)$, and then

$$\lim_{\gamma\to 0}\frac{X_q^{11}(\theta)}{X_q^{11}(-\theta)} \sim e^{\frac{6i}{\gamma}\int_{-\infty}^{\theta} d\theta' \log(X^{11}(\theta'))}.x^{3/2}e^{\frac{8\pi^2 i}{3\gamma}},$$

the factor $x^{3/2}$ is cancelled in physical S-matrix elements. There are the additional constant factors

$$e^{\frac{iA}{\gamma}} = e^{\frac{6i}{\gamma}\int_{-\infty}^{0} d\theta' \log(X^{11}(\theta'))}e^{\frac{8\pi^2 i}{3\gamma}},$$

which have yet to be computed explicitly, and are related to the number of bound states in the channel under consideration. We have explicitly

$$\lim_{\gamma\to 0} S_{11}^{11}(\theta) = \lim_{\gamma\to 0} e^{\theta/2}x^{-3/2}\frac{X_q^{11}(\theta)}{X_q^{11}(-\theta)} = e^{\frac{iA}{\gamma}+\frac{i6}{\gamma}\int_0^{\theta} d\theta' \log(X^{11}(\theta'))}.$$

Here, the lower indices denote certain topological charges, and we recognize this as the correct semi-classical behaviour.

Observe that in the solution for $v^{11}(\theta)$ (9.11), we trivially have the unitarity equation (9.8) for $v^{11}(\theta)$. There is clearly a single-valuedness problem in the square-roots of the quantum dilogarithms in the denominator of (9.12). This will turn out not to be a problem, the combination $X_q^{11}(\theta)/X_q^{11}(-\theta)$, which makes up physical S-matrix elements is truly single-valued, as shall be discussed below.

We calculate the solution $v^{12}(\theta)$ by crossing $v^{11}(\theta)$, using equation (9.9):

$$\begin{aligned} X_q^{11}(x^{-1}q^{-1}) &= \frac{S_{q^{-3}}(e^{i\pi}x^{-3})}{\sqrt{S_{q^{-3}}(e^{i\pi}x^{-3}q^{-2})S_{q^{-3}}(e^{-i\pi}x^{-3}q^{-4})}} \\ &= \sqrt{(1+e^{-i\pi}x^{-3}q^{-1})}\frac{S_{q^{-3}}(e^{i\pi}x^{-3})}{\sqrt{S_{q^{-3}}(e^{i\pi}x^{-3}q^{-2})S_{q^{-3}}(e^{-i\pi}x^{-3}q^{2})}}, \end{aligned}$$

$$X_q^{11}(xq)^{-1} = \frac{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q^4)S_{q^{-3}}(e^{-i\pi}x^3q^2)}}{S_{q^{-3}}(e^{i\pi}x^3q^6)}$$

$$= \frac{\sqrt{(1+e^{i\pi}x^3q)}}{(1-x^3q^3)}\frac{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q^{-2})S_{q^{-3}}(e^{-i\pi}x^3q^2)}}{S_{q^{-3}}(e^{i\pi}x^3)}.$$

We define

$$\begin{aligned} X^{12}_q(x) &= \frac{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q^{-2})S_{q^{-3}}(e^{-i\pi}x^3q^2)}}{S_{q^{-3}}(e^{i\pi}x^3)} \\ &= \frac{\sqrt{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{\frac{2i\pi}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{-\frac{2i\pi}{3}})}}{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta})}, \end{aligned} \tag{9.13}$$

noting that in the same way as for the $X^{11}_q(\theta)$ case, $X^{12}_q(\theta)$ reproduces the correct semi-classical limit based on the time delay $X^{12}(\theta)$. We then have for $X^{11}_q(x)$

$$X^{11}_q(x^{-1}q^{-1}) = \sqrt{(1+e^{-i\pi}x^{-3}q^{-1})}X^{12}_q(x^{-1})^{-1}$$

$$X^{11}_q(xq)^{-1} = \frac{\sqrt{(1+e^{i\pi}x^3q)}}{(1-x^3q^3)}X^{12}_q(x).$$

We note that the classical 'crossing' equation for $X^{ij}(\theta)$ which in this case reads $X^{11}(i\pi-\theta) = X^{21}(-\theta)^{-1}$, allows us to organise $X^{11}_q(i\pi-\theta)$ as $X^{21}_q(-\theta)^{-1}$ times some extraneous x-dependent factors obtained after repeatedly using the quantum dilogarithm property (6.5).

In the general case this classical crossing property $X^{ij}(i\pi-\theta) = X^{\bar{i}j}(-\theta)^{-1}$, will always guarantee that $X^{ij}_q(i\pi-\theta)$ can be written as $X^{\bar{i}j}_q(-\theta)^{-1}$ times some extraneous factors. The combination $X^{ij}_q(\theta)/X^{ij}_q(-\theta)$ will cross into $X^{\bar{i}j}_q(\theta)/X^{\bar{i}j}_q(-\theta)$ times these additional factors. These extraneous factors are important in the method.

Hence we have the solution

$$\begin{aligned} v^{11}(i\pi-\theta) = v^{21}(\theta) = v^{12}(\theta) &= -\frac{qe^{-\theta/2}e^{i\pi/2}}{(1-x^{-3}q^{-1})}\frac{X^{11}_q(x^{-1}q^{-1})}{X^{11}_q(xq)} \\ &= -\frac{qe^{-\theta/2}e^{i\pi/2}}{(1-x^{-3}q^{-1})}\frac{\sqrt{(1+e^{-i\pi}x^{-3}q^{-1})(1+e^{i\pi}x^3q)}}{(1-x^3q^3)}\frac{X^{12}_q(\theta)}{X^{12}_q(-\theta)} \\ &= \frac{e^{-\theta/2}}{(x^{-3/2}q^{-3/2}-x^{3/2}q^{3/2})}\frac{X^{12}_q(\theta)}{X^{12}_q(-\theta)}. \end{aligned} \tag{9.14}$$

We see that the prefactor in this solution for $v^{12}(\theta)$, $(x^{3/2}q^{3/2} - x^{-3/2}q^{-3/2})^{-1}$, which we derived from crossing $X^{11}_q(\theta)$ is precisely the prefactor which is needed to ensure the unitarity condition (9.8) for $v^{12}(\theta)$. We have also implicitly computed the prefactor $(1-x^3q^2)^{-1}$ in $v^{11}(\theta)$, (9.11), since this factor when crossed was cancelled by the factor

$$\sqrt{(1+e^{-i\pi}x^{-3}q^{-1})(1+e^{i\pi}x^3q)},$$

which was generated by the crossing of $X_q^{11}(\theta)/X_q^{11}(-\theta)$. Therefore all the prefactors in $v^{11}(\theta)$ and $v^{12}(\theta)$ which are required for unitarity are generated by the crossing.

To summarise, the postulated solutions are the following:

$$\begin{aligned} v^{11}(\theta) &= \frac{-qe^{\theta/2}}{(1-x^3q^2)}\frac{X_q^{11}(\theta)}{X_q^{11}(-\theta)}, \\ v^{12}(\theta)=v^{21}(\theta) &= \frac{e^{-\theta/2}}{(x^{-3/2}q^{-3/2}-x^{3/2}q^{3/2})}\frac{X_q^{12}(\theta)}{X_q^{12}(-\theta)}. \end{aligned} \tag{9.15}$$

Where

$$\begin{aligned} X_q^{11}(\theta) &= \frac{S_{q^{-3}}(e^{i\pi}x^3q^3)}{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q)S_{q^{-3}}(e^{-i\pi}x^3q^{-1})}} \\ X_q^{12}(\theta) &= \frac{\sqrt{S_{q^{-3}}(e^{i\pi}x^3q^{-2})S_{q^{-3}}(e^{-i\pi}x^3q^2)}}{S_{q^{-3}}(e^{i\pi}x^3)}. \end{aligned} \tag{9.16}$$

Several remarks about these solutions are in order. We have shown that they formally satisfy the equations (9.8) and (9.9) and that they have the correct semi-classical behaviour given by the time delays $X^{11}(\theta)$ and $X^{12}(\theta)$ in the limit $\gamma\to 0$. But it is not yet clear that the solutions are single-valued. An obvious obstacle to single-valuedness is the presence of the square-roots of the quantum dilogarithms in $X_q^{ij}(\theta)$, the zeroes and poles of the square-rooted quantum dilogarithms would apparently appear as branch points in the S-matrix, which are, of course, unacceptable. To overcome this problem, we use the fact that (6.11)

$$S_\mu(w)S_\mu(w^{-1})=e^{\frac{i(\log w)^2}{4\mu}}.e^{-\frac{i}{12}(\mu+\frac{\pi^2}{\mu})}, \tag{9.17}$$

(the second constant factor is irrelevant), and note that in the denominator of $X_q^{11}(\theta)$, the arguments of the quantum dilogarithms are of the form x^3w and x^3w^{-1}, for $w=e^{i\pi}q$. Hence in the combination $X_q^{11}(\theta)/X_q^{11}(-\theta)$, which is required for the S-matrix, the square-rooted factors combine to give

$$\sqrt{\frac{S_{q^{-3}}(e^{i\pi}x^{-3}q)S_{q^{-3}}(e^{-i\pi}x^{-3}q^{-1})}{S_{q^{-3}}(e^{i\pi}x^3q)S_{q^{-3}}(e^{-i\pi}x^3q^{-1})}}=\frac{C(\theta)}{S_{q^{-3}}(e^{i\pi}x^3q)S_{q^{-3}}(e^{-i\pi}x^3q^{-1})},$$

where $C(\theta)$ is a single-valued function, related to the first factor in (9.17). We see that the obstacle to single-valuedness has disappeared since there are no square-roots.

We can similarly repeat this discussion on single-valuedness for the other S-matrix $S^{12}(\theta)$, $X_q^{12}(\theta)$ is given by equation (9.16), and it again satisfies the crucial property that the square-rooted quantum dilogarithms in the numerator have arguments of the form x^3w and x^3w^{-1}, where $w=e^{i\pi}q^{-2}$.

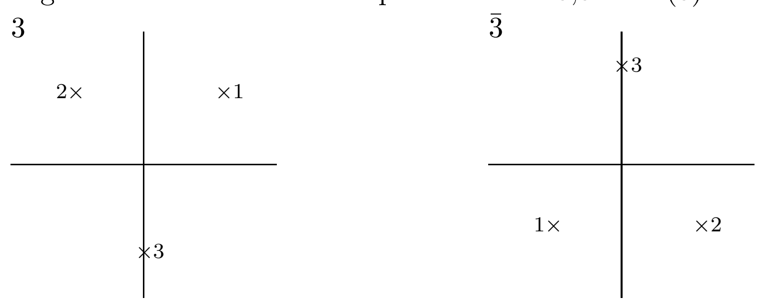
Another remark that we wish to make about the solution concerns the choice of the coupling constant independent phases $e^{\pm i\pi}$ in the arguments of the quantum dilogarithms.

These disappear from the semi-classical limit when we use the duality transformation (6.8) to take the limit $\gamma \to 0$. They obviously must be of the form $e^{i\pi+2\pi ni}$, where n is an integer, since only then do we reproduce the correct factors by crossing which are required for unitarity (9.8), with a relative minus sign in each factor. Suppose we choose the first phase in $X_q^{11}(\theta)$, (9.12), in the denominator to be $e^{i\pi+2\pi ni}$, the second phase in the denominator is then fixed to be $e^{-(i\pi+2\pi ni)}$, as required for single-valuedness, and then the phase of the quantum dilogarithm in the numerator is then fixed by requiring the soliton bootstrap to work, see below. We then cross this solution over to $X_q^{12}(\theta)$ and so all the coupling constant independent phases in $X_q^{12}(\theta)$ are completely fixed. This shows that the solution is completely determined up to the arbitary integer n. The choice in the integer n moves the positions of the simple poles, due to the fusing solitons, around. This particular choice, $n=0$, is chosen so that a pole lies precisely where we would expect it, at the same place as the simple pole in the classical $X^{ij}(\theta)$, which leads to the classical fusion of the two-soliton solution into a third single-soliton solution, at $\theta = iU_{ij}^{\bar{k}}$, the fusing angle corresponding to the process $i+j \to k$.

As in sine-Gordon, it is always possible to write the solution with $n \neq 0$, as the solution with $n=0$, multiplied by the CDD factors, obtained by repeated application of (6.5) in the dual description.

The poles

We label the weights of the fundamental representations 3,$\bar{3}$ of $su(3)$ as follows:



The poles of

$$\frac{X_q^{11}(\theta)}{X_q^{11}(-\theta)} = C(\theta).\frac{S_{q^{-3}}(e^{i\pi}x^3q^3)S_{q^{-3}}(e^{-i\pi}x^3q^{-3})}{S_{q^{-3}}(e^{i\pi}x^3q)S_{q^{-3}}(e^{-i\pi}x^3q^{-1})} \tag{9.18}$$

where $C(\theta)$ does not contain poles or zeroes, from the double infinite product expansion of $S_{q^{-3}}(w)$ (6.9), are at the following places:

The poles arising from the initially square-rooted factors (from fusing solitons)

$$\begin{aligned} \theta &= -i\Big(\pi\frac{(6l+2)}{3} + \frac{\gamma}{6}(k+1)\Big), \\ \theta &= -i\Big(\pi\frac{(6l+4)}{3} + \frac{\gamma}{6}(k)\Big). \end{aligned} \tag{9.19}$$

Poles arising from the initially non-square-rooted factors

$$\theta = i\Big(2\pi l + \frac{\gamma}{6}(k+1)\Big),$$

$$\theta \;=\; i\Big(2\pi(l+1)+\frac{\gamma}{6}k\Big), \tag{9.20}$$

and we have zeroes at precisely minus these positions, where $k, l = 0, 1, 2, \ldots$. On the physical strip these poles lie at precisely $\theta = i\frac{\gamma}{6}(k+1)$. There are no poles on the physical strip from the first set of factors, and the zeroes lie at $\theta = i\Big(\frac{2\pi}{3}+\frac{\gamma}{6}(k+1)\Big)$. There are no zeroes on the physical strip from the second set of factors.

Now consider actual S-matrix elements, rather than the solutions $v^{11}(\theta)$ and $v^{12}(\theta)$. From explicit computations of the matrices $R^{11}(x)$ and $R^{12}(x)$, we have, (the lower indices denoting topological charges, which are labelled by the diagrams above)

$$S^{11}_{11}(\theta) = e^{\theta/2}x^{-3/2}\frac{X^{11}_q(\theta)}{X^{11}_q(-\theta)}, \tag{9.21}$$

and so the only poles on the physical strip are at $\theta = i\frac{\gamma(k+1)}{6}$, $k = 0, 1, 2, \ldots$. We interpret these as crossed $S^{12}_{11}(\theta)$ breather poles, so that $S^{12}_{11}(\theta)$ has direct channel breather poles at $\theta = i\pi - i\frac{\gamma(k+1)}{6}$, $k = 0, 1, 2, \ldots$. We do not expect to see a pole due to the soliton fusion, $1+1 \to 2$, since the topological charges, or weights 1 plus 1 in the 3 representation do not give a weight in $\bar{3}$. We expect, of course, to see such a pole in $S^{11}_{12\to 21}(\theta)$ (transmission), since the weight 1 in the representation 3 plus the weight 2 in 3 is equal to the weight 3 in the representation $\bar{3}$. Now

$$S^{11}_{12\to 21}(\theta) = \frac{\sinh(\frac{6\pi\theta}{\gamma})}{\sinh(\frac{6\pi\theta}{\gamma}-\frac{4\pi^2 i}{\gamma})}x^{-3/2}e^{\theta/2}\frac{X^{11}_q(\theta)}{X^{11}_q(-\theta)}. \tag{9.22}$$

The prefactor contributes zeroes at $\theta = \frac{im\gamma}{6}$, $m \in \mathcal{Z}$, and poles at

$$\theta = \frac{2\pi i}{3} + \frac{in\gamma}{6}, \quad n \in \mathcal{Z}. \tag{9.23}$$

Therefore the poles on the physical strip in $S^{11}_{12}(\theta)$ lie at $\theta = \frac{2\pi i}{3} - \frac{in\gamma}{6}$, $n = 0, 1, 2, \ldots$. The poles with $n > 0$ in (9.23) are cancelled by the zeroes of $X^{11}_q(\theta)/X^{11}_q(-\theta)$, and the poles of $X^{11}_q(\theta)/X^{11}_q(-\theta)$ at $\theta = ik\gamma/6$, $k = 0, 1, \ldots$, are cancelled by the zeroes of the prefactor at $\theta = i\frac{m\gamma}{6}$.

The pole on the physical strip at $\theta = \frac{2\pi i}{3}$ is due to the prefactor multiplying $X^{11}_q(\theta)/X^{11}_q(-\theta)$, which is designed to ensure that the unitarity condition is satisfied, and not $X^{11}_q(\theta)/X^{11}_q(-\theta)$ itself. The factors due to the fusing solitons (the ones that are square-rooted in $X^{11}_q(\theta)$) do not apparently introduce poles onto the physical strip. However the prefactors required for unitarity arise from crossing $X^{11}_q(\theta)/X^{11}_q(-\theta)$ as explained earlier. In fact for each $\sqrt{S(wx^3)S(w^{-1}x^3)}$ in the denominator of $X^{ij}_q(\theta)$ we pick up a factor $(1+wx^3)^{-1}$ for the prefactor in $v^{ij}(\theta)$, and for the factor $S(wx^3)$ in the denominator of $X^{ij}_q(\theta)$ we also pick up a factor $(1+wx^3)^{-1}$. This means that the simple poles in $X^{ij}(\theta)$, which were previously explained in terms of the explicit classical fusing of solitons, are also at the same place in the S-matrix, provided they are not cancelled by zeroes in $X^{ij}_q(\theta)/X^{ij}_q(-\theta)$, as discussed in section 3.

The soliton bootstrap

We must check the internal consistency of the proposed result (9.15). We must check that the simple poles interpreted as solitons do in fact generate S-matrix elements consistent with the solutions so far derived. The method that is available for doing this is the so-called bootstrap method, as discussed in section 4, which allows us to compute new S-matrices from a given S-matrix. We can calculate the S-matrix for scattering the state interpreted as the bound state in the given S-matrix (specified by the pole) with one of the original states in the given S-matrix.

The ground state soliton case

We check this first for the ground state soliton, given by the pole at $\theta = \frac{2\pi i}{3}$, in $S^{11}(\theta)$. We interpret it as pole due to the soliton of species 2, corresponding to the fusing $1+1 \to 2$.

We must check that the following matrix equation holds

$$S^{12}(\theta) = (I \otimes S^{11}(\theta + \frac{i\pi}{3}))(S^{11}(\theta - \frac{i\pi}{3}) \otimes I)\Big|_{3\otimes\bar{3}}, \tag{9.24}$$

where the notation means that we restrict the right-hand side which is valued in the space $3 \otimes 3 \otimes 3$ to the space $3 \otimes \bar{3}$. By using equation (6.11), we write $v^{11}(\theta)$ and $v^{12}(\theta)$ in the form

$$v^{11}(\theta) = \frac{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{-i\pi})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{i\pi})}{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{-2\pi i+\theta}e^{\frac{i\pi}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{-\frac{i\pi}{3}})}, \tag{9.25}$$

and

$$v^{12}(\theta) = \frac{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{-\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{\theta}e^{\frac{2\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^{-3}e^{-2\pi i+\theta})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta})}, \tag{9.26}$$

and we strip off any possible constant factors, and factors of the form $e^{\kappa\theta}$, in front of these expressions.

In order to check the bootstrap we compute

$$v^{11}(\theta + \frac{i\pi}{3})v^{11}(\theta - \frac{i\pi}{3}).$$

In order to be able to cancel the $S_{\tilde{q}^{-3}}(w)$ when we compute this, it is helpful if we replace *all* $S_{\tilde{q}^{-3}}(\tilde{q}^{-3}w)$ with $S_{\tilde{q}^{-3}}(\tilde{q}^{3}w)$, and in the process pick up a factor of the form $(1 + e^{\theta}e^{\frac{2\pi i p}{h}})$, each time we make the replacement. So we write

$$\begin{aligned} v^{11}(\theta) &= \frac{(1+e^{\theta}e^{-\frac{i\pi}{3}})}{(1+e^{\theta}e^{-i\pi})}\cdot\frac{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{-i\pi})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{i\pi})}{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{-2\pi i+\theta}e^{\frac{i\pi}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{-\frac{i\pi}{3}})} \\ &= B^{11}(\theta)^{-1}\tilde{v}^{11}(\theta), \end{aligned} \tag{9.27}$$

where

$$B^{11}(\theta)^{-1} = \frac{(1+e^{\theta}e^{-\frac{i\pi}{3}})}{(1+e^{\theta}e^{-i\pi})},$$

and similarly

$$v^{12}(\theta) = \frac{(1+e^{\theta})}{(1+e^{\theta}e^{\frac{2\pi i}{3}})}\cdot\frac{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{-\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta}e^{\frac{2\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{-2\pi i+\theta})S_{\tilde{q}^{-3}}(\tilde{q}^{3}e^{\theta})}$$

$$= \quad B^{12}(\theta)^{-1}\tilde{v}^{12}(\theta), \tag{9.28}$$

where

$$B^{12}(\theta)^{-1} = \frac{(1+e^{\theta})}{(1+e^{\theta}e^{\frac{2\pi i}{3}})}.$$

These equations serve to define $\tilde{v}^{ij}(\theta)$ and $B^{ij}(\theta)$. $B^{ij}(\theta)$ can be considered as 'half' of the classical $X^{ij}(\theta)$, in the sense that

$$X^{ij}(\theta) = \text{const.}B^{ij}(\theta).B^{ij}(-\theta). \tag{9.29}$$

For example, in $su(3)$,

$$B^{11}(\theta)B^{11}(-\theta) = -e^{\frac{i\pi}{3}}\frac{(1+e^{\theta}e^{-i\pi})(1+e^{\theta}e^{i\pi})}{(1+e^{\theta}e^{-\frac{i\pi}{3}})(1+e^{\theta}e^{\frac{i\pi}{3}})} = -e^{\frac{i\pi}{3}}X^{11}(\theta).$$

In the same way that the bootstrap equation is satisfied by the classical $X^{ij}(\theta)$, equation (3.10), the bootstrap equation, in this example, is also satisfied by only this 'half' of $X^{ij}(\theta)$, namely $B^{ij}(\theta)$.

For $su(3)$, we check the bootstrap equation for $B^{ij}(\theta)$

$$B^{11}(\theta+\frac{i\pi}{3})B^{11}(\theta-\frac{i\pi}{3}) = B^{12}(\theta),$$

$$\frac{(1+e^{\theta}e^{-\frac{2\pi i}{3}})}{(1+e^{\theta})}\frac{(1+e^{\theta}e^{-\frac{4\pi i}{3}})}{(1+e^{\theta}e^{-\frac{2\pi i}{3}})} = \frac{(1+e^{\theta}e^{\frac{2\pi i}{3}})}{(1+e^{\theta})}.$$

Now the bootstrap equation for the full classical $X^{ij}(\theta)$, equation (3.10), guarantees that the bootstrap is satisfied by $\tilde{v}^{ij}(\theta)$ (although we shall see this in detail below), except for factors of the form $(1-x^h q^p)$, obtained by altering the argument of $S_{\tilde{q}^{-3}}(w)$ by $e^{2\pi i}$. We have already dealt with the factors of the form $(1+e^{\theta}e^{\frac{2\pi i p}{h}})$, which are required when we alter the argument by $\tilde{q}^{-6}$. Now

$$\begin{aligned}
\tilde{v}^{11}(\theta+\frac{i\pi}{3})\tilde{v}^{11}(\theta-\frac{i\pi}{3}) &= \frac{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{-\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{\frac{4\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{-2\pi i+\theta}e^{\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta})}\frac{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{-\frac{4\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{\frac{2\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{-2\pi i+\theta})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{-\frac{2\pi i}{3}})} \\
&= \frac{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{\frac{4\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{-2\pi i+\theta})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta})} \\
&= \frac{1}{1-x^3q^{-1}}\frac{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{\frac{2\pi i}{3}})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta}e^{-\frac{2\pi i}{3}})}{S_{\tilde{q}^{-3}}(\tilde{q}^3e^{-2\pi i+\theta})S_{\tilde{q}^{-3}}(\tilde{q}^3e^{\theta})} \\
&= \frac{1}{1-x^3q^{-1}}\tilde{v}^{12}(\theta),
\end{aligned} \tag{9.30}$$

and hence

$$v^{11}(\theta+\frac{i\pi}{3})v^{11}(\theta-\frac{i\pi}{3}) = \frac{1}{1-x^3q^{-1}}v^{12}(\theta). \tag{9.31}$$

The factor that we have picked up here, $(1 - x^3q^{-1})^{-1}$, is accounted for by the discrepancy between the normalisation of

$$(I \otimes R^{11}(xq^{-1/3}))(R^{11}(xq^{1/3}) \otimes I)\Big|_{3\otimes\bar{3}},$$

and $R^{12}(x)$ defined with no extraneous factors, that is with no overall factors multiplying each element of the matrix. $R^{11}(x)$ must cross into $R^{12}(x)$ with no extraneous factors, since we have imposed $v^{11}(i\pi - \theta) = v^{12}(\theta)$. By defining $R^{12}(x)$ like this, to be in its lowest factorised form, we guarantee that this crossing condition holds. It is possible to show that

$$(x^{3/2}q^{-1/2} - x^{-3/2}q^{1/2})R^{12}(x) = (I \otimes R^{11}(xq^{-1/3}))(R^{11}(xq^{1/3}) \otimes I)\Big|_{3\otimes\bar{3}}. \tag{9.32}$$

Hence the correct bootstrap equation for S, equation (9.24), follows.

10 The general solution

Properties and definition of $X_q^{ij}(\theta)$

We shall discover that the solution to the soliton S-matrix is

$$v^{ij}(\theta) = f(x).\frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)},$$

where we shall shortly define $X_q^{ij}(\theta)$, and where $f(x)$ consists of products of the form $(1 - x^hq^p)^{-1}$, and are related to the unitarity condition since $v^{ij}(\theta)v^{ji}(-\theta) = f(x).f(x^{-1})$.

For $c(i) - c(j) \geq 0$, we define

$$X_q^{ij}(\theta) = \prod_{p=0}^{h-1} S_{q^{-h}}\left(e^{is_p\pi}x^hq^{-\frac{(c(i)-c(j))}{2}}q^{-2p+h}\right)^{\frac{\gamma_i\cdot\sigma^p\gamma_j}{2}}, \tag{10.1}$$

for

$$x = e^{\frac{4\pi\theta}{\gamma}}, \qquad q = e^{-\frac{4\pi^2 i}{\gamma}}.$$

With $s_p = \pm 1$, $e^{is_p\pi} = e^{\pm i\pi}$ are certain phases to be defined below.

For $c(i) - c(j) = -2$, we define

$$X_q^{ij}(\theta) = \prod_{p=1}^{h} S_{q^{-h}}\left(e^{is_p\pi}x^hq^{-\frac{(c(i)-c(j))}{2}}q^{-2p+h}\right)^{\frac{\gamma_i\cdot\sigma^p\gamma_j}{2}}. \tag{10.2}$$

The differences in the ranges of p in the two definitions are designed to ensure that powers of q in the arguments of $S_{q^{-h}}$ are between and including h and $-h$. This in turn ensures that poles due to fusing solitons and breathers are in the correct places. The mod $2\pi i$ difference in the positions of the poles obtained by replacing p with $p+h$ in (10.1) and (10.2) is important, since the S-matrices are not $2\pi i$ periodic.

Observe that for $c(i) - c(j) = 0$ and $i \neq j$, $\gamma_i.\gamma_j = 0$, so that the $p = 0$ term of (10.1) and the $p = h$ term of (10.2) vanish. Hence (10.1) and (10.2) are the same in this case. If $i = j$ however, there is an important difference, and we must take (10.1).

The semi-classical limit

Reverting to the dual picture (6.8), with $c(i) - c(j) \geq 0$, we have

$$X_q^{ij}(\theta) = \prod_{p=0}^{h-1} S_{\tilde{q}^{-h}}\left(\tilde{q}^{-s_p h} e^{\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p}{h} - i\pi}\right)^{\frac{\gamma_i \cdot \sigma^p \gamma_j}{2}} \tag{10.3}$$

where

$$\tilde{q}^{-h} = e^{\frac{\gamma i}{4h}}.$$

It is easy to see that[2]

$$\lim_{\gamma \to 0} X_q^{ij}(\theta) = \text{const}.e^{\frac{2hi}{\gamma}\int_0^{\theta} d\theta' \left(\log\left(X^{ij}(\theta')\right)^{1/2}\right)}.$$

Hence

$$\lim_{\gamma \to 0} \frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)} = \text{const}.e^{\frac{2hi}{\gamma}\int_0^{\theta} d\theta' \log(X^{ij}(\theta'))}, \tag{10.4}$$

where we have used the fact that $X^{ij}(\theta) = X^{ij}(-\theta)$.

Single-valuedness

$$\frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)} \quad \text{is single valued.}$$

Proof

We only need to consider the factors with $\gamma_i \cdot \sigma^p \gamma_j = -1$, if $\gamma_i \cdot \sigma^p \gamma_j = 1$, we cross the result so that these factors are in the denominator.

For $\gamma_i \cdot \sigma^p \gamma_j = -1$ and

$$p' = h - p + \frac{c(j) - c(i)}{2},$$

then [15], $\gamma_i \cdot \sigma^{p'} \gamma_j = -1$. In fact all the values of p, such that $\gamma_i \cdot \sigma^p \gamma_j = -1$, pair up in this way. If $p = p'$, then $\gamma_i \cdot \sigma^p \gamma_j = -2$ (when $i = \bar{j}$), or $\gamma_i \cdot \sigma^p \gamma_j = 0$.

Hence for

$$T_p(\theta) = S_{\tilde{q}^{-h}}\left(\tilde{q}^{-s_p h} e^{\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p}{h} - i\pi}\right)^{-1/2} S_{\tilde{q}^{-h}}\left(\tilde{q}^{-s_p' h} e^{\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p'}{h} - i\pi}\right)^{-1/2},$$

the argument of the second quantum dilogarithm is

$$\tilde{q}^{-s_{p'} h} e^{\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p'}{h} - i\pi} =$$

$$\tilde{q}^{-s_{p'} h} e^{\theta} e^{-\frac{i\pi}{2h}(c(i)-c(j))} e^{-\frac{2\pi i p}{h} + i\pi}.$$

We choose the phases, $s_p = \pm 1$, in such a way so that $s_p = -s_{p'}$. Hence

$$T_p(\theta) = S_{\tilde{q}^{-h}}(e^{\theta} w)^{-1/2} S_{\tilde{q}^{-h}}(e^{\theta} w^{-1})^{-1/2},$$

[2]Bearing in mind the sine-Gordon and $su(3)$ cases.

with

$$w=\tilde{q}^{-s_ph}e^{\frac{i\pi}{2h}(c(i)-c(j))}e^{\frac{2\pi ip}{h}-i\pi}.$$

This completes the proof because (6.11)

$$S_{\tilde{q}^{-h}}(e^{\theta}w^{-1})^{-1/2}=(\text{single valued factor}).S_{\tilde{q}^{-h}}(e^{-\theta}w)^{1/2}.$$

Then

$$\frac{T_p(\theta)}{T_p(-\theta)}=(\text{single valued factor}).\frac{1}{S_{\tilde{q}^{-h}}(e^{\theta}w)S_{\tilde{q}^{-h}}(e^{\theta}w^{-1})},$$

which is single-valued.

$X_q^{ij}(\theta)/X_q^{ij}(-\theta)$ is made up of products of $T_p(\theta)/T_p(-\theta)$ with $p\geq\frac{h}{2}+\frac{c(j)-c(i)}{4}$, say, and the corresponding crossed factors, when $\gamma_i\cdot\sigma^p\gamma_j=1$, and the already single-valued factors when $\gamma_i\cdot\sigma^p\gamma_j=\pm2$.

The phases $e^{is_p\pi}$

We define the set of integers $A^{ij}_{->}$ to be the set p such that $\gamma_i\cdot\sigma^p\gamma_j=-1$ and with $p\geq\frac{h}{2}+\frac{c(j)-c(i)}{4}$. If $c(i)-c(j)\geq0$, we take $p\in\{0,\ldots,h-1\}$, otherwise $p\in\{1,\ldots,h\}$.

Each p, such that $\gamma_i\cdot\sigma^p\gamma_j=-1$, pairs with another value of p, p', where $p'=h-p+\frac{c(j)-c(i)}{2}$, and we have excluded the smaller of $\{p,p'\}$ from $A^{ij}_{->}$. We similarly define the set of integers $A^{ij}_{-<}$ by excluding the larger of $\{p,p'\}$.

By crossing we can also define the set $A^{ij}_{+>}$ in a similar way but for those p with $\gamma_i\cdot\sigma^p\gamma_j=1$ and the larger of the two values $\{p,p'\}$ such that $\gamma_i\cdot\sigma^p\gamma_j=\gamma_i\cdot\sigma^{p'}\gamma_j$. And in the obvious way, we also define $A^{ij}_{+<}$. In fact $A^{ij}_{+<}$ and $A^{\bar{\imath}j}_{->}$ are related by $p\in A^{ij}_{+<}$ if and only if $p+\frac{h}{2}+\frac{c(i)-c(\bar{\imath})}{4}\in A^{\bar{\imath}j}_{->}$, where mod h is implicitly not understood. $A^{ij}_{+>}$ and $A^{\bar{\imath}j}_{-<}$ are related by $p\in A^{ij}_{+>}$ if and only if $p-\frac{h}{2}+\frac{c(i)-c(\bar{\imath})}{4}\in A^{\bar{\imath}j}_{-<}$, see equation (10.6).

We choose the phase $e^{is_p\pi}$ as follows:

$$\begin{aligned}s_p=-1,&\qquad\text{for }p\in A^{ij}_{->},\\ s_p=+1,&\qquad\text{for }\gamma_i\cdot\sigma^p\gamma_j=-2.\end{aligned}\tag{10.5}$$

For single-valuedness we must demand that $s_p=+1$, for $p\in A^{ij}_{-<}$. The remaining values of p give $\gamma_i\cdot\sigma^p\gamma_j=1,2$, and the remaining s_p are fixed by the crossing symmetry, since when we cross an $X_q^{ij}(\theta)$, those factors with $\gamma_i\cdot\sigma^p\gamma_j=1,2$ in $X_q^{ij}(\theta)$, are in the denominator of $X_q^{\bar{\imath}j}(\theta-i\pi)$, and are therefore treated by the first set of rules (10.5). They are fixed as

$$\begin{aligned}p'\in A^{\bar{\imath}j}_{->},&\qquad s_p=-1,\\ p'\in A^{\bar{\imath}j}_{-<},&\qquad s_p=+1,\\ \gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j=-2,&\qquad s_p=+1.\end{aligned}$$

where $p'=p+\frac{h}{2}+\frac{c(i)-c(\bar{\imath})}{4}$ mod h. This is because [15], $\gamma_i=-\sigma^{-\frac{h}{2}-\frac{c(i)-c(\bar{\imath})}{4}}\gamma_{\bar{\imath}}$, and therefore

$$\gamma_i\cdot\sigma^p\gamma_j=-\gamma_{\bar{\imath}}\cdot\sigma^{p+\frac{h}{2}+\frac{c(i)-c(\bar{\imath})}{4}}\gamma_j.\tag{10.6}$$

Note that these rules for s_p mean that for p in the lower half of its range and $\gamma_i\cdot\sigma^p\gamma_j=\pm1$, $s_p=-\gamma_i\cdot\sigma^p\gamma_j$. In the upper half of the range, $s_p=\gamma_i\cdot\sigma^p\gamma_j$. If $\gamma_i\cdot\sigma^p\gamma_j=\pm2$, then $s_p=1$.

Symmetry: $X_q^{ij}(\theta) = X_q^{ji}(\theta)$

We refer to the proof of the symmetry $X^{ij}(\theta) = X^{ji}(\theta)$ in the classical case, see [9]. We again use $\gamma_i \cdot \sigma^p \gamma_j = \gamma_j \cdot \sigma^{p'} \gamma_i$, with

$$2p + \frac{c(i) - c(j)}{2} = 2p' + \frac{c(j) - c(i)}{2}.$$

If $c(i) - c(j) = 0$, $p = p'$ and the result is trivially true.
If $c(i) - c(j) = 2$, $p + 1 = p'$, and

$$\begin{aligned} X_q^{ij}(\theta) &= \prod_{p=0}^{h-1} S_{q^{-h}} \left(e^{is_p \pi} x^h q^{-\frac{(c(i)-c(j))}{2}} q^{-2p+h} \right)^{\frac{\gamma_i \cdot \sigma^p \gamma_j}{2}} \\ &= \prod_{p'=1}^{h} S_{q^{-h}} \left(e^{is_{p'-1} \pi} x^h q^{-\frac{(c(j)-c(i))}{2}} q^{-2p'+h} \right)^{\frac{\gamma_j \cdot \sigma^{p'} \gamma_i}{2}} \\ &= X_q^{ji}(\theta). \end{aligned} \tag{10.7}$$

Note that the phases $e^{is_p \pi}$ come out correctly, since in this case $A^{ij}_{\pm >} + 1 = A^{ji}_{\pm >}$ and $A^{ij}_{\pm <} + 1 = A^{ji}_{\pm <}$, where there is no implicit understanding that we take p mod h. If $c(i) - c(j) = -2$, we swap i and j in the proof above.

The crossing property

For

$$v^{ij}(\theta) = A^{ij} e^{c^{ij}\theta} \frac{1}{\prod_{p \in A^{ij}_{->}} \left(1 - x^h q^{-\frac{c(i)-c(j)}{2}} q^{2h-2p} \right)} \frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)}, \tag{10.8}$$

with the aditional factor $(1 - x^h q^h)^{-1}$ on the right-hand side, if $i = \bar{\jmath}$. Then we have $v^{ji}(i\pi - \theta) = v^{\bar{\imath} j}(\theta)$. Here A^{ij} and c^{ij} are certain constants.

We also trivially have $v^{ij}(\theta) = v^{ji}(\theta)$.

Proof

We refer to the proof of the classical crossing property, equation (3.5, in section 3, $X^{ij}(\theta + i\pi) = X^{\bar{\imath} j}(\theta)^{-1}$. For $c(i) - c(j) \geq 0$,

$$X_q^{ij}(\theta) = \prod_{p=0}^{h-1} S_{q^{-h}} \left(e^{is_p \pi} x^h q^{-\frac{(c(i)-c(j))}{2}} q^{-2p+h} \right)^{\frac{\gamma_i \cdot \sigma^p \gamma_j}{2}}.$$

From [15]

$$\gamma_i = -\sigma^{-\frac{h}{2} - \frac{c(i) - c(\bar{\imath})}{4}} \gamma_{\bar{\imath}},$$

and we compute $X_q^{ij}(\theta + i\pi)$,

$$X_q^{ij}(\theta + i\pi) = \prod_{p=0}^{h-1} S_{q^{-h}} \left(e^{is_p \pi} x^h q^{-\frac{(c(i)-c(j))}{2}} q^{-2p} \right)^{-\frac{\gamma_{\bar{\imath}} \cdot \sigma^{p + \frac{h}{2} + \frac{c(i)-c(\bar{\imath})}{4}} \gamma_j}{2}}.$$

Let $p' = p + \frac{h}{2} + \frac{c(i)-c(\bar{\imath})}{4} = p + r$, where r is an integer and is positive,

$$= \prod_{p'=r}^{h-1+r} S_{q^{-h}}\left(e^{is_p\pi} x^h q^{-\frac{(c(\bar{\imath})-c(j))}{2}} q^{-2p'+h}\right)^{-\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j}{2}}.$$

For $c(\bar{\imath}) - c(j) \geq 0$,

$$= \prod_{p'=0}^{h-1} S_{q^{-h}}\left(e^{is_p\pi} x^h q^{-\frac{(c(\bar{\imath})-c(j))}{2}} q^{-2p'+h}\right)^{-\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j}{2}} . \prod_{p=0}^{r-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{-2p}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}}$$

$$= X_q^{\bar{\imath}j}(\theta)^{-1} . \prod_{p=0}^{r-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{-2p}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}}. \qquad (10.9)$$

Note that the s_p have been defined so that $s_p^{ij} = s_{p'}^{\bar{\imath}j}$, and the limits of the range of p are appropriate for the value of $c(i) - c(j)$ (see the definitions of $A_{\pm>}^{ij}$ and $A_{\pm<}^{ij}$, the integers in these sets have been defined to lie in the appropriate ranges) and the range of p' appropriate for the value of $c(\bar{\imath}) - c(j)$. We also compute $X_q^{ij}(\theta - i\pi)$

$$X_q^{ij}(\theta - i\pi) = \prod_{p=0}^{h-1} S_{q^{-h}}\left(e^{is_p\pi} x^h q^{-\frac{(c(i)-c(j))}{2}} q^{-2p+2h}\right)^{-\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p+\frac{h}{2}+\frac{c(i)-c(\bar{\imath})}{4}}\gamma_j}{2}}$$

$$= \prod_{p'=r}^{h-1+r} S_{q^{-h}}\left(e^{is_p\pi} x^h q^{-\frac{(c(\bar{\imath})-c(j))}{2}} q^{-2p'+3h}\right)^{-\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j}{2}}. \qquad (10.10)$$

For $c(\bar{\imath}) - c(j) \geq 0$,

$$= \prod_{p'=0}^{h-1} S_{q^{-h}}\left(e^{is_p\pi} x^h q^{-\frac{(c(\bar{\imath})-c(j))}{2}} q^{-2p'+h}\right)^{-\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j}{2}} . \prod_{p=r}^{h-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{2h-2p}\right)^{\frac{-\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}}$$

$$= X_q^{\bar{\imath}j}(\theta)^{-1} \prod_{p=r}^{h-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{2h-2p}\right)^{\frac{-\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}}. \qquad (10.11)$$

Combining these two together

$$\frac{X_q^{ij}(i\pi - \theta)}{X_q^{ij}(-(i\pi - \theta))} = \frac{X_q^{\bar{\imath}j}(\theta)}{X_q^{\bar{\imath}j}(-\theta)} \prod_{p=r}^{h-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{2h-2p}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}} \prod_{p=0}^{r-1}\left(1 - x^{-h} q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{-2p}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}},$$

letting $p' = h - p + \frac{c(j)-c(\bar{\imath})}{2}$ in the second product,

$$= \frac{X_q^{\bar{\imath}j}(\theta)}{X_q^{\bar{\imath}j}(-\theta)} \prod_{p=r}^{h-1}\left(1 - x^h q^{-\frac{c(\bar{\imath})-c(j)}{2}} q^{2h-2p}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p}\gamma_j}{2}} \prod_{p'=h+\frac{c(j)-c(\bar{\imath})}{2}-(r-1)}^{h+\frac{c(j)-c(\bar{\imath})}{2}}\left(1 - x^{-h} q^{\frac{c(\bar{\imath})-c(j)}{2}} q^{2p'-2h}\right)^{\frac{\gamma_{\bar{\imath}}\cdot\sigma^{p'}\gamma_j}{2}}$$

$$= \quad Ae^{c\theta}\frac{X_q^{\bar{\imath}j}(\theta)}{X_q^{\bar{\imath}j}(-\theta)}\frac{\prod_{p\in A^{\bar{\imath}j}_{+>}}\left(1-x^{-h}q^{\frac{c(\bar{\imath})-c(j)}{2}}q^{-(2h-2p)}\right)}{\prod_{p\in A^{\bar{\imath}j}_{->}}\left(1-x^{h}q^{-\frac{c(\bar{\imath})-c(j)}{2}}q^{2h-2p}\right)}, \tag{10.12}$$

if $j=\bar{\imath}$, there is the additional factor $(1-x^{-h})$ on the right-hand side of equation (10.12), if $j=i$ then there is the additional factor $(1-x^hq^h)^{-1}$.

Putting $f(\theta)$ as the numerator of the products in (10.12),

$$f(\theta)=\prod_{p\in A^{\bar{\imath}j}_{+>}}\left(1-x^{-h}q^{\frac{c(\bar{\imath})-c(j)}{2}}q^{-(2h-2p)}\right).$$

If $p\in A^{\bar{\imath}j}_{+>}$, then $p-\frac{h}{2}-\frac{c(i)-c(\bar{\imath})}{4}\in A^{ij}_{-<}$, and thus

$$p'=\frac{3h}{2}-p+\frac{c(i)-c(\bar{\imath})}{4}+\frac{c(j)-c(i)}{2}\in A^{ij}_{->}.$$

Then

$$\begin{aligned} f(i\pi-\theta) &= \prod_{p'\in A^{ij}_{->}}\left(1-x^{h}q^{\frac{c(\bar{\imath})-c(j)}{2}}q^{-h+3h-2p'-(\frac{c(\bar{\imath})+c(i)}{2})+c(j)}\right) \\ &= \prod_{p'\in A^{ij}_{->}}\left(1-x^{h}q^{-\frac{c(i)-c(j)}{2}}q^{2h-2p'}\right). \end{aligned} \tag{10.13}$$

If $j=\bar{\imath}$, the additional factor, $(1-x^{-h})$, should be included in $f(\theta)$, when $\theta\to i\pi-\theta$, this factor becomes $(1-x^hq^h)$. We see that this is the same as the factor present in the denominator of equation (10.12), if $i=j$.

Hence we have established that for

$$v^{ij}(\theta)=A^{ij}e^{c^{ij}\theta}\frac{1}{\prod_{p\in A^{ij}_{->}}\left(1-x^{h}q^{-\frac{c(i)-c(j)}{2}}q^{2h-2p}\right)}\frac{X_q^{ij}(\theta)}{X_q^{ij}(-\theta)}, \tag{10.14}$$

with the aditional factor, $(1-x^hq^h)^{-1}$, when $i=\bar{\jmath}$, then we have the crossing condition $v^{ij}(i\pi-\theta)=v^{\bar{\imath}j}(\theta)$.

The proof, giving the same formula for v^{ij} (10.14), is modified slightly for the case $c(\bar{\imath})-c(j)<0$, and then subsequently for the case $c(i)-c(j)<0$, but remains largely unchanged. These cases are left to the reader.

We note that for $p\in A^{ij}_{->}$, $p'=p-(\frac{c(j)-c(i)}{2})\in A^{ji}_{->}$, where mod h is not implicit in this last equation. Therefore

$$\prod_{p\in A^{ij}_{->}}\left(1-x^{h}q^{-\frac{c(i)-c(j)}{2}}q^{2h-2p}\right)=\prod_{p'\in A^{ji}_{->}}\left(1-x^{h}q^{-\frac{c(j)-c(i)}{2}}q^{2h-2p'}\right).$$

Since we have already shown that $X_q^{ij}(\theta)=X_q^{ji}(\theta)$, we have trivially established the symmetry $v^{ij}(\theta)=v^{ji}(\theta)$. The correct crossing condition then follows for $v^{ij}(\theta)$, $v^{ji}(i\pi-\theta)=v^{\bar{\imath}j}(\theta)$.

11 Comparison with the known R-matrices

We compare the proposed solutions (10.8) with known R-matrices, which have been computed in [32]. As outlined in section 5, it is possible to compute $R^{ij}(\theta)$ in terms of the projection matrices that project onto the representations in the decomposition of $V_i \otimes V_j$ into representations. It is only possible to do this when the multiplicity is one for each of the representations in the decomposition. This does not happen for all i, and j, in general. However the method always works for the fundamental elements $R^{11}(x)$ (with slight alterations for D_n, so we must use $R^{nn}(x)$ if n is odd, and $R^{nn}(x)$, $R^{n,n-1}(x)$, $R^{n-1,n-1}(x)$, and $R^{11}(x)$, if n is even), and we shall compare our conjectures in these cases. Then by application of the soliton bootstrap we should be able to recover all the elements $S^{ij}(\theta)$, not explicitly needing all the other matrices $R^{ij}(\theta)$.

Comparison with the fundamental element $R^{11}(\theta)$

Here we shall find that the fundamental element $v^{11}(\theta)$ always has the correct unitarity condition given directly by the prefactors from (10.8), recall that these were derived from the crossing symmetry.

A_n:

From [32], equation (4.17), with

$$< n >= \frac{x - q^n}{1 - xq^n},$$

in the homogeneous gradation,

$$R^{11}(x) = P_0 + < 2 > P_1,$$

where $P_0 + P_1 = 1$, and P_1 projects onto the fundamental representation V_2. We move to the principal gradation by replacing x with x^h and conjugating by the matrix $\sigma(x)$. We also change the normalisation by multiplying by $(1 - x^h q^2)$[3].We must adopt this normalisation, by clearing all the denominators of $< n > \ldots < m >$, in all the R-matrices which are to be checked. This is an important point, since if we had adopted the normalisation initially given, we would have had the unitarity equation $R^{11}(x)R^{11}(x^{-1}) = 1$, we would then have had to remove the prefactor in $v^{11}(\theta)$, given by equation (10.8), and hence we would have generated factors on crossing $v^{11}(\theta)$, these compensate for factors needed when we cross the matrix $R^{11}(x)$, depending on the normalisation chosen for the crossed matrix. These different ways of dealing with the factors are equivalent, since it is only a question of normalisation, but it seems clearer to arrange the normalisation so the crossing condition on the v is always $v^{ij}(i\pi - \theta) = v^{\bar{i}j}(\theta)$. This is not the approach used in [2].

So, in the principal gradation, we have

$$R^{11}_P(x) = \sigma P_0 \sigma^{-1}(1 - x^h q^2) + \sigma P_1 \sigma^{-1}(x^h - q^2),$$

note that at the pole in $v^{11}(\theta)$, corresponding to the fusing $1+1 \to 2$, $\theta = \frac{2\pi i}{h}$, or $(1 - x^h q^2) = 0$ so that $R^{11}_P(x)$ projects onto the the fundamental representation V_2.

[3]It is crucial to do this because only then will this matrix be able to cross precisely into another matrix with no extraneous factors, which must be normalized in the same way, so that all the denominators in $< n > \ldots < m >$ are cleared. By inspection this is the only way of getting the matrices to cross into each other.

We then compute

$$\begin{aligned} R_P^{11}(x)R_P^{11}(x^{-1}) &= \sigma(P_0(1-x^hq^2)(1-x^{-h}q^2)+P_1(x^h-q^2)(x^{-h}-q^2))\sigma^{-1} \\ &= (1-x^hq^2)(1-x^{-h}q^2)\sigma(P_0+P_1)\sigma^{-1} \\ &= (1-x^hq^2)(1-x^{-h}q^2).1 \end{aligned} \tag{11.1}$$

Now the prefactor in the A_n theories in $v^{11}(\theta)$, given by equation (10.8), is (from the fusing $1+1 \to 2$) [4]

$$\frac{1}{1-x^hq^2},$$

so that

$$v^{11}(x)v^{11}(x^{-1}) = \frac{1}{(1-x^hq^2)(1-x^{-h}q^2)},$$

and the above calculation for $R_P^{11}(x)R_P^{11}(x^{-1})$ shows that $S^{11}(\theta)S^{11}(-\theta)=1$, as required.
D_n, $h=2n-2$,

For n even we need to check $R^{11}(x)$, and we may as well also check $R^{11}(x)$ for n odd. So for all n, from [32], equation (4.49), (noting that there is a printing error in passing from (4.48) to (4.49), making equation (4.49) slightly incorrect), we have

$$R^{11}(x) = P_0 + <2> P_1 + <2><h> P_2, \tag{11.2}$$

P_1 projects onto V_2 and P_2 onto the trivial representation (for breathers). Hence, after adopting the special normalisation needed for crossing, as has been discussed above,

$$R_P^{11}(x) = \sigma P_0\sigma^{-1}(1-x^hq^2)(1-x^hq^h)+\sigma P_1\sigma^{-1}(x^h-q^2)(1-x^hq^h)+\sigma P_2\sigma^{-1}(x^h-q^2)(x^h-q^h),$$

and

$$\begin{aligned} R_P^{11}(x)R_P^{11}(x^{-1}) &= (1-x^hq^2)(1-x^hq^h).(1-x^{-h}q^2)(1-x^{-h}q^h)\sigma(P_0+P_1+P_2)\sigma^{-1} \\ &= (1-x^hq^2)(1-x^hq^h).(1-x^{-h}q^2)(1-x^{-h}q^h).1 \end{aligned} \tag{11.3}$$

The fusings are $1+1 \to 0,2$, and the prefactor for $v^{11}(\theta)$, from (10.8), is

$$\frac{1}{(1-x^hq^2)(1-x^hq^h)},$$

and this agrees with $R_P^{11}(x)R_P^{11}(x^{-1})$.

For n even we need to check $R^{nn}(x)$ and $R^{n-1n}(x)$, from (4.44) and (4.45) of [32],

$$R^{nn}(x) = P_0' + \sum_{a=1}^{\frac{n}{2}} \prod_{i=1}^{a} <4i-2> P_{n-2a}$$

[4] The most direct route to finding the prefactors is to use the fusing angle for the appropriate fusing, the prefactor must place a pole at this fusing angle. These angles are available from affine Toda particle S-matrix data[12, 11].

where here P_{n-2a} projects onto the space V_{n-2a}, and the fusings are $n+n \to 0, 2, 4, \ldots, n-2$, and the prefactor to $v^{nn}(\theta)$ is, from equation (10.8),

$$\frac{1}{(1-x^h q^2)(1-x^h q^6)\ldots(1-x^h q^h)},$$

this agrees exactly with the factors derived from $R_P^{nn}(x)R_P^{nn}(x^{-1})$, provided we normalise $R_P^{nn}(x)$ in the manner that has already been discussed.

For n even, we also have

$$R^{n,n-1}(x) = P_0' + \sum_{a=1}^{\frac{n}{2}-1} \prod_{i=1}^{a} < 4i > P_{n-2a-1},$$

and again P_{n-2a-1} projects onto V_{n-2a-1}, the fusings are $n+(n-1) \to 1, 3, 5, \ldots n-3$, and hence the prefactor to $v^{n,n-1}(\theta)$, from equation (10.8), is

$$\frac{1}{(1-x^h q^4)(1-x^h q^8)\ldots(1-x^h q^{h-2})},$$

this agrees with the $R^{nn-1}(x)$.

For n odd, we must check $R^{nn}(x)$, this is the fundamental element in that case.

$$R^{nn}(x) = P_0' + \sum_{a=1}^{\frac{n-1}{2}} \prod_{i=1}^{a} < 4i-2 > P_{n-2a}$$

where here P_{n-2a} projects onto the space V_{n-2a}, the fusings are $n+n \to 1, 3, 5, \ldots n-2$, and the prefactor to $v^{nn}(\theta)$ is

$$\frac{1}{(1-x^h q^2)(1-x^h q^6)(1-x^h q^{10})\ldots(1-x^h q^{h-2})},$$

this agrees with $R^{nn}(x)$.
E_6:

Equation (4.55) from [32] is

$$R^{11}(x) = P_0 + < 2 > P_1 + < 2 >< 8 > P_2,$$

P_1 projects onto V_3, and P_2 projects onto V_6. The fusings are $1+1 \to 2, 6$, implying the prefactor to $v^{11}(\theta)$

$$\frac{1}{(1-x^h q^2)(1-x^h q^8)},$$

this agrees with $R^{11}(x)$.
E_7: Equation (89) of [33], gives

$$R^{11}(x) = P_0 + < 2 > P_1 + < 2 >< 10 > P_2 + < 2 >< 10 >< 18 > P_3,$$

where P_1 projects onto V_4, P_2 onto V_2 and P_3 onto V_0. The fusings are $1+1 \to 0, 2, 4$, and these imply the prefactor to $v^{11}(\theta)$

$$\frac{1}{(1-x^h q^2)(1-x^h q^{10})(1-x^h q^{18})}.$$

Unfortunately the case E_8 has not been worked out in [32, 33], and so we cannot check the fundamental element of E_8. However in principle, given the work [32, 33], it should not be too difficult to find $R^{11}(x)$, also in this case, and check to see if it agrees with our $v^{11}(\theta)$.

Comparison with other elements

A_n:

All fundamental representations V_i and V_j are amenable to the method and, for $N = \min(j, n-i)$,

$$
\begin{aligned}
R^{ij}(x) = P_0 + &< 2+i-j > P_1 + < 2+i-j >< 4+i-j > P_2 \\
&+ < 2+i-j >< 4+i-j >< 6+i-j > P_3 + \cdots \\
&+ < 2+i-j >< 4+i-j > \ldots < 2N+i-j > P_N,
\end{aligned} \tag{11.4}
$$

where P_N projects onto V_{i+j}. We observe that

$$R^{i1}(\theta) = P_0 + < 1+i > P_1,$$

implying after the appropriate normalising that

$$R_P^{11}(x) R_P^{11}(x^{-1}) = (1 - x^h q^{1+i})(1 - x^{-h} q^{1+i}) 1,$$

and this agrees with the prefactor to $v^{1i}(\theta)$, namely $(1 - x^h q^{1+i})^{-1}$.

However, if there are more than two representations in the decomposition of $V_i \otimes V_j$, we do not get agreement with the prefactor in $v^{ij}(\theta)$, and further corrections to $v^{ij}(\theta)$ will be needed. For example

$$R^{22}(x) = P_0 + < 2 > P_1 + < 2 >< 4 > P_2,$$

from $R_P^{22}(x) R_P^{22}(x^{-1})$ this implies the prefactor to $v^{22}(\theta)$, $((1 - x^h q^2)(1 - x^h q^4))^{-1}$, but the prefactor is, from (10.8) $(1 - x^h q^4)^{-1}$, and so we require a 'correction' $(1 - x^h q^2)^{-1}$ to $v^{22}(\theta)$. We must then separately cross over this extra factor when we compute $v^{22}(i\pi - \theta) = v^{\bar{2}2}(\theta) = v^{n-1,2}(\theta)$. We will now see whether an extra factor is required for $v^{n-1,2}(\theta)$, and if it is consistent with the crossed factor previously found for $v^{22}(\theta)$.

$$R^{n-1,2}(\theta) = P_0 + < h-2 > P_1 + < h-2 >< h > P_2,$$

the prefactor already present in $v^{n-1,2}(\theta)$ is $(1 - x^h q^h)^{-1}$, and so an extra factor is needed, and must be $(1 - x^h q^{h-2})^{-1}$. When crossed, $x \to x^{-1} q^{-1}$, this extra factor becomes

$$\frac{1}{1 - x^{-h} q^{-2}} = \frac{-x^h q^2}{(1 - x^h q^2)}. \tag{11.5}$$

The extra $-x^h q^2$ is not important, and this (11.5) is essentially the extra prefactor needed for $v^{22}(\theta)$.

These corrections are consistent with the bootstrap, but this is harder to see because of the differences in the normalisation of $R^{ij}(x)$, defined in its lowest factorised form, and $R^{ij}(x)$

defined by fusion, as discussed for $su(3)$. However, in some cases, there is no difference in this normalisation, for example in A_6,

$$(1 \otimes R^{12}(\theta + \frac{i\pi}{7}))(R^{12}(\theta - \frac{i\pi}{7}) \otimes 1)|_{V_2 \otimes V_2} = R^{22}(\theta),$$

where here, $R^{ij}(\theta)$ must have the special normalisation given by clearing the denominators of $< n > .. < m >$ in equation (11.4). By counting powers of $(1 - x^h q^p)$ on both sides of this equation, we see that the normalisations agree. Both sides have a power two.

We check the bootstrap equation for $v^{ij}(\theta)$, in this specific case:

From (12.2),

$$\begin{aligned}
\tilde{v}^{12}(\theta) &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\frac{6\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{6\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{4\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{4\pi i}{7}})} \\
\tilde{v}^{22}(\theta) &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-i\pi}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{i\pi})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{3\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{3\pi i}{7}})},
\end{aligned} \tag{11.6}$$

and

$$\begin{aligned}
&\tilde{v}^{12}(\theta + \frac{i\pi}{7}) \tilde{v}^{12}(\theta - \frac{i\pi}{7}) \\
&= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{i\pi}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{5\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{5\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{3\pi i}{7}})} \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\frac{5\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\pi i})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{3\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{5\pi i}{7}})} \\
&= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\pi i}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\pi i})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{3\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{3\pi i}{7}})} . \frac{1}{1 - x^h q^2} \\
&= \tilde{v}^{22}(\theta) \frac{1}{1 - x^h q^2}.
\end{aligned} \tag{11.7}$$

This indicates that we must correct $v^{22}(\theta)$ by $(1 - x^h q^2)^{-1}$, and this, of course, agrees with what we said before. We can read off these additional corrections required for $v^{ij}(\theta)$, directly from (11.4). For simplicity we restrict ourselves to A_6, although it is immediately obvious how to extend this to A_n. We write a for the factor $(1 - x^h q^a)^{-1}$.

$i \backslash j$	1	2	3	4	5	6
1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
2	–	2	3	4	5	$\surd$
3	–	–	2 4	3 5	4	$\surd$
4	–	–	–	2 4	3	$\surd$
5	–	–	–	–	2	$\surd$
6	–	–	–	–	–	$\surd$

Table 1: Corrections to $v^{ij}(\theta)$ for A_6

For the A_n theories, the normalisation of the fused R-matrices can be determined completely [3], and we can show that this table of corrections is consistent with the bootstrap on the $v^{ij}(\theta)$. This will be sketched in section 13. For the D and E series, there will be similar corrections, which have not yet been determined. The reason for this is that, so far, we cannot determine the normalisation of the fused R-matrices. The trick that was used for A_n, cannot be repeated for the other algebras.

12 The poles of the general solution

$v^{ij}(\theta)$, as defined by equation (10.8), has simple poles on the physical strip at the following places:
For $p \in A^{ij}_{->}$,

$$\theta = i\Big(2\pi - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(i) - c(j)) - \frac{\gamma n}{2h}\Big), \qquad n = 0, 1, 2, \ldots$$

If $i = \bar{\jmath}$, we have breather poles at

$$\theta = i\pi - i\frac{\gamma n}{2h}, \qquad n = 1, 2, 3, \ldots$$

If $i = j$, then we have crossed breather poles at

$$\theta = i\frac{\gamma n}{2h}, \qquad n = 1, 2, 3, \ldots$$

Proof
With the specific choices of the phases $e^{is_p\pi}$, that have already been specified, we have

$$\begin{aligned}\frac{X^{ij}_q(\theta)}{X^{ij}_q(-\theta)} = Ae^{c\theta} \prod_{p\in A^{ij}_{->}} & S_{\tilde{q}^{-h}}(\tilde{q}^h e^{\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p}{h} - i\pi})^{-1} S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{\theta} e^{-\frac{i\pi}{2h}(c(i)-c(j))} e^{-\frac{2\pi i p}{h} + i\pi})^{-1} \\ & \times \prod_{p\in A^{\bar{\imath}j}_{->}} S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-i\pi+\theta} e^{\frac{i\pi}{2h}(c(\bar{\imath})-c(j))} e^{\frac{2\pi i p}{h} - i\pi}) S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{i\pi+\theta} e^{-\frac{i\pi}{2h}(c(\bar{\imath})-c(j))} e^{-\frac{2\pi i p}{h} + i\pi}) \\ & \qquad \times \frac{S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{\theta} e^{-i\pi})}{S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{-\theta} e^{-i\pi})}\bigg|_{\text{if } i=j} \frac{S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{-\theta})}{S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{\theta})}\bigg|_{\text{if } i=\bar{\jmath}}, \end{aligned} \qquad (12.1)$$

where A and c are certain, irrelevant, constants.

This form (12.1) for $X^{ij}_q(\theta)/X^{ij}_q(-\theta)$ is an alternative method of seeing the crossing property (10.12).

We can also write

$$\begin{aligned} v^{ij}(\theta) = A'e^{c'\theta} \prod_{p\in A^{ij}_{->}} & S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{i\pi}{2h}(c(i)-c(j))} e^{\frac{2\pi i p}{h} - i\pi})^{-1} S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{\theta} e^{-\frac{i\pi}{2h}(c(i)-c(j))} e^{-\frac{2\pi i p}{h} + i\pi})^{-1} \\ \times \prod_{p\in A^{\bar{\imath}j}_{->}} & S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-i\pi+\theta} e^{\frac{i\pi}{2h}(c(\bar{\imath})-c(j))} e^{\frac{2\pi i p}{h} - i\pi}) S_{\tilde{q}^{-h}}(\tilde{q}^{-h} e^{i\pi+\theta} e^{-\frac{i\pi}{2h}(c(\bar{\imath})-c(j))} e^{-\frac{2\pi i p}{h} + i\pi}) \end{aligned}$$

$$\times \frac{S_{\tilde{q}^{-h}}(\tilde{q}^{-h}e^{\theta}e^{-i\pi})S_{\tilde{q}^{-h}}(\tilde{q}^{h}e^{\theta}e^{i\pi})}{1}\Bigg|_{\text{if } i=j} \frac{1}{S_{\tilde{q}^{-h}}(\tilde{q}^{-h}e^{-2\pi i+\theta})S_{\tilde{q}^{h}}(\tilde{q}^{h}e^{\theta})}\Bigg|_{\text{if } i=\bar{j}}, \tag{12.2}$$

Consider the contribution in (12.1) from $p \in A^{ij}_{->}$, from (6.9), we have simple poles at

$$\theta = -i\Big(\pi(2k) + \frac{2\pi p}{h} + \frac{\pi}{2h}(c(i)-c(j)) + \frac{\gamma}{4h}(2l)\Big),$$

and at

$$\theta = -i\Big(\pi(2k+2) - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(i)-c(j)) + \frac{\gamma}{4h}(2l+2)\Big),$$

and zeroes at precisely minus these positions, where $k, l = 0, 1, 2, \ldots$ On the physical strip, there are no poles, and zeroes at

$$\theta = i\Big(\frac{2\pi p}{h} + \frac{\pi}{2h}(c(i)-c(j)) + \frac{\gamma}{2h}l\Big),$$

and

$$\theta = i\Big(2\pi - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(i)-c(j)) + \frac{\gamma}{2h}(l+1)\Big).$$

In fact since $p \geq \frac{h}{2} + \frac{c(j)-c(i)}{4}$, the zeroes from the first set are actually not on the physical strip.

For

$$v^{ij}(\theta) = A^{ij}e^{c^{ij}\theta}\frac{1}{\prod_{p\in A^{ij}_{->}}\Big(1 - x^h q^{-\frac{c(i)-c(j)}{2}}q^{2h-2p}\Big)}\frac{X^{ij}_q(\theta)}{X^{ij}_q(-\theta)}.$$

The prefactor contributes simple poles at

$$e^{\frac{4\pi\theta h}{\gamma}}e^{2\pi i n} = e^{-\frac{4\pi^2 i}{\gamma}\frac{c(i)-c(j)}{2}}e^{-\frac{4\pi^2 i}{\gamma}(2p-2h)},$$

or

$$\theta = i\Big(2\pi - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(i)-c(j)) - \frac{\gamma}{2h}n\Big), \qquad n \in \mathcal{Z}.$$

Therefore the poles with $n = -1, -2, \ldots$ are cancelled by the zeroes, and we are left with poles at

$$\theta = i\Big(2\pi - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(i)-c(j)) - \frac{\gamma}{2h}n\Big), \qquad n = 0, 1, 2, \ldots$$

in $v^{ij}(\theta)$.

In fact we have chosen the phases $e^{is_p\pi}$ in such a way so that there is a pole at $n = 0$ corresponding to the ground state fusing soliton. This is at precisely the same position as the pole present in the classical $X^{ij}(\theta)$, see section 3.

Now consider the contribution from $p \in A^{\bar{\imath}j}_{->}$ in (12.1), these poles are essentially in the crossed position of the poles from $v^{\bar{\imath}j}(\theta)$.

On the physical strip the poles are at

$$\begin{aligned}\theta &= -i\pi + i\Big(\frac{2\pi p}{h} + \frac{\pi}{2h}(c(\bar{\imath})-c(j)) + \frac{\gamma}{2h}l\Big)\\ &= i\pi - i\Big(2\pi - \frac{2\pi p}{h} - \frac{\pi}{2h}(c(\bar{\imath})-c(j)) - \frac{\gamma}{2h}l\Big), \qquad l = 0, 1, 2, \ldots\end{aligned} \tag{12.3}$$

If $i = \bar{\jmath}$, there are no poles in (12.1) on the physical strip, but there are zeroes at

$$\theta = i\Big(\pi(2k+1) + \frac{\gamma}{2h} l\Big),$$

for $k, l = 0, 1, 2, \ldots$, so that there is a zero at $\theta = i\pi$. From the prefactor $(1 - x^h q^h)^{-1}$ in the equation (10.8) for $v^{ij}(\theta)$, there are poles at

$$\theta = i\pi - i\frac{\gamma}{2h} n, \qquad n \in \mathcal{Z}.$$

The zero at $\theta = i\pi$ cancels the pole at $\theta = i\pi$ and we therefore have breather poles at

$$\theta = i\pi - i\frac{\gamma}{2h} n, \qquad n = 1, 2, 3, \ldots$$

If $i = j$, we pick up crossed breather poles in $v^{ij}(\theta)$ at

$$\theta = i\frac{\gamma}{2h}(l+1), \qquad l = 0, 1, 2, \ldots$$

with no contribution from the prefactor.

13 The general soliton bootstrap

Starting from $v^{11}(\theta)$, given by equation (12.2), we follow through the bootstrap, comparing the result obtained with the $v^{ij}(\theta)$, defined by (12.2), with the additional corrections of the form $(1 - x^h q^p)^{-1}$ discussed in the A_n case. Following the $su(3)$ case, discussed in section 9, the coefficients $B^{ij}(\theta)$ are defined by turning all the $\tilde{q}^{-h}$'s in the arguments of the $S_{\tilde{q}^{-h}}(w)$'s into $\tilde{q}^h$.

Define the block

$$(x)_\theta = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})}.$$

We start with A_6.

A_6, $\qquad c(1) = -1$,

p, q	0	1	2	3	4	5	6	7	$h = 7$
$\gamma_1 \cdot \sigma^p \gamma_1$	2	-1	0	0	0	0	-1		$p' = 7 - p$
$\lambda_1 \cdot \sigma^q \gamma_1$	-1	0	0	0	0	0	1		$q' = 6 - q$
s_p	1	1	0	0	0	0	-1		

We calculate $B^{ij}(\theta)$ by taking the values of p, such that $s_p = 1$, from the table above. For each value of p, we include the factor

$$(1 + e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(j)}{2}) - i\pi} e^\theta)^{\mathrm{sign}(\gamma_i \cdot \sigma^p \gamma_j) 1} \tag{13.1}$$

in $B^{ij}(\theta)$.

So

$$B^{11}(\theta) = \frac{(1 + e^{i\pi} e^\theta)}{(1 + e^{-\frac{5\pi i}{7}} e^\theta)}.$$

For $B^{12}(\theta)$, we have

$$A_6, \qquad\qquad c(1)=-1,\ c(2)=1,$$

p,q	0	1	2	3	4	5	6	7	$h=7$
$\gamma_1\cdot\sigma^p\gamma_2$		1	-1	0	0	0	-1	1	$p'=8-p$
$\lambda_1\cdot\sigma^q\gamma_2$		-1	0	0	0	0	1	0	$q'=7-q$
s_p		-1	1				-1	1	

and therefore, from the values of p such that $s_p=1$,

$$B^{12}(\theta)=\frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{4\pi i}{7}}e^{\theta})},$$

we check the bootstrap, from the fusing $1+1\to 2$, as follows,

$$\begin{aligned}B^{11}(\theta-\frac{i\pi}{7})B^{11}(\theta+\frac{i\pi}{7}) &= \frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{6\pi i}{7}}e^{\theta})}\frac{(1+e^{-\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{4\pi i}{7}}e^{\theta})}\\ &= \frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{4\pi i}{7}}e^{\theta})}=B^{12}(\theta).\end{aligned} \tag{13.2}$$

In this case the bootstrap checks out, and no corrections are needed.

Now, from the fusing $1+1\to 2$, we compute $B^{22}(\theta)$.

$$\begin{aligned}B^{12}(\theta+\frac{i\pi}{7})B^{12}(\theta-\frac{i\pi}{7}) &= \frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{5\pi i}{7}}e^{\theta})}\\ &= \frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}\frac{\sinh(\frac{\theta}{2}-\frac{2\pi i}{14})}{\sinh(\frac{\theta}{2}+\frac{2\pi i}{14})}\\ &= B^{22}(\theta)(2)_\theta^{-1},\end{aligned} \tag{13.3}$$

but

$$B^{22}(\theta)=\frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}.$$

Hence we have to 'correct' $v^{22}(\theta)$, as given in equation (12.2), by $(2)_\theta$, since $v^{ij}(\theta)=B^{ij}(\theta)^{-1}\tilde{v}^{ij}(\theta)$. This introduces another pole at $\theta=\frac{2\pi i}{7}$ on the physical strip. Observe that there is a double pole in this position in the particle S-matrix $S^{22}(\theta)=\{1\}_\theta\{3\}_\theta$, which is not explained in terms of fusings, but as a Landau singularity [11].

We continue with the bootstrap on $B^{ij}(\theta)$. For the fusing $1+2\to 3$,

$$B^{13}(\theta)=\frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})},$$

$$B^{11}(\theta-\frac{2\pi i}{7})B^{12}(\theta+\frac{i\pi}{7}) = \frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-i\pi}e^{\theta})}\frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}$$

$$= \frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}$$

$$= B^{13}(\theta), \tag{13.4}$$

so the bootstrap checks out, without introducing any extraneous factors.

We compute $B^{32}(\theta)$ from the fusing $1+2 \to 3$,

$$B^{32}(\theta) = \frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{2\pi i}{7}}e^{\theta})},$$

$$B^{12}(\theta + \frac{2\pi i}{7})B^{22}(\theta - \frac{i\pi}{7}) = \frac{(1+e^{\frac{8\pi i}{7}}e^{\theta})}{(1+e^{-\frac{2\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{4\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{4\pi i}{7}}e^{\theta})}{(1+e^{-\frac{6\pi i}{7}}e^{\theta})}$$

$$= \frac{(1+e^{\frac{6\pi i}{7}}e^{\theta})}{(1+e^{-\frac{2\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{4\pi i}{7}}e^{\theta})}{(1+e^{-\frac{4\pi i}{7}}e^{\theta})} = B^{32}(\theta)(3)_{\theta}^{-1}, \tag{13.5}$$

here we have already included the $(2)_{\theta}^{-1}$ in $B^{22}(\theta)$.

Consider the fusing $1+3 \to 4$, after including the correction $(3)_{\theta}^{-1}$ in $B^{32}(\theta)$,

$$B^{32}(\theta + \frac{\pi i}{7})B^{12}(\theta - \frac{3\pi i}{7}) = \frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{3\pi i}{7}}e^{\theta})}{(1+e^{-i\pi}e^{\theta})}$$

$$= \frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{3\pi i}{7}}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})} = B^{42}(\theta)(4)_{\theta}^{-1}, \tag{13.6}$$

where

$$B^{42}(\theta) = \frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{\pi i}{7}}e^{\theta})}.$$

However, we must demand that (with the corrections) $v^{32}(i\pi - \theta) = v^{\bar{3}2}(\theta) = v^{42}(\theta)$, and we must check that the corrections cross over appropriately. It is easy to show $(3)_{i\pi-\theta} = (4)_{\theta}$, as required.

Continuing with the bootstrap with the fusing $1+2 \to 3$, we compute $B^{33}(\theta)$,

$$B^{33}(\theta) = \frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{\pi i}{7}}e^{\theta})}.$$

But

$$B^{23}(\theta + \frac{\pi i}{7})B^{13}(\theta - \frac{2\pi i}{7}) = \frac{(1+e^{i\pi}e^{\theta})}{(1+e^{-\frac{\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{5\pi i}{7}}e^{\theta})}{(1+e^{-\frac{3\pi i}{7}}e^{\theta})}\frac{(1+e^{\frac{3\pi i}{7}}e^{\theta})}{(1+e^{-\frac{5\pi i}{7}}e^{\theta})}$$

$$= \quad B^{33}(\theta)(2)_\theta^{-1}(4)_\theta^{-1}. \tag{13.7}$$

For the fusing $3+1 \to 4$, we compute $B^{34}(\theta)$,

$$B^{34}(\theta) = \frac{(1+e^{\frac{6\pi i}{7}}e^\theta)}{(1+e^\theta)}$$

and

$$B^{33}(\theta - \frac{\pi i}{7})B^{31}(\theta + \frac{3\pi i}{7}) \quad = \quad \frac{(1+e^{\frac{6\pi i}{7}}e^\theta)}{(1+e^{-\frac{2\pi i}{7}}e^\theta)}\frac{(1+e^{\frac{4\pi i}{7}}e^\theta)}{(1+e^{-\frac{4\pi i}{7}}e^\theta)}\frac{(1+e^{\frac{2\pi i}{7}}e^\theta)}{(1+e^{-\frac{6\pi i}{7}}e^\theta)}\frac{(1+e^{\frac{8\pi i}{7}}e^\theta)}{(1+e^\theta)}$$

$$= \quad B^{34}(\theta)(3)_\theta^{-1}(5)_\theta^{-1}. \tag{13.8}$$

We proceed in this manner and find all the corrections in Table 2b. We summarise the $B^{ij}(\theta)$ for A_6 in Table 2a. The notation for this is that we write

$$\frac{a\ b \ldots d}{e\ f \ldots g}$$

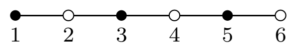
for

$$\frac{(1+e^{\frac{ia}{h}}e^\theta)(1+e^{\frac{ib}{h}}e^\theta)\ldots(1+e^{\frac{id}{h}}e^\theta)}{(1+e^{\frac{ie}{h}}e^\theta)(1+e^{\frac{if}{h}}e^\theta)\ldots(1+e^{\frac{ig}{h}}e^\theta)}.$$

Note the remarkable fact that Table 2b is of the same form as Table 1, so that the pole on the physical strip due to a $(p)_\theta$ from Table 2b is doubled up by a pole from the factor $(1-x^h q^p)^{-1}$, from Table 1.

In tables 3a,b to 6a,b, we present the cases D_5, E_6, E_7 and E_8 respectively, computed by following through the bootstrap. We show both the $B^{ij}(\theta)$ and the corrections.

$A_6, \qquad h = 7,$



$B^{ij}(\theta)$: $i\backslash j$	1	2	3	4	5	6
1	$\frac{7}{-5}$	$\frac{6}{-4}$	$\frac{5}{-3}$	$\frac{4}{-2}$	$\frac{3}{-1}$	$\frac{2}{0}$
2	–	$\frac{7}{-3}$	$\frac{6}{-2}$	$\frac{5}{-1}$	$\frac{4}{0}$	$\frac{3}{-1}$
3	–	–	$\frac{7}{-1}$	$\frac{6}{0}$	$\frac{5}{-1}$	$\frac{4}{-2}$
4	–	–	–	$\frac{7}{-1}$	$\frac{6}{-2}$	$\frac{5}{-3}$
5	–	–	–	–	$\frac{7}{-3}$	$\frac{6}{-4}$
6	–	–	–	–	–	$\frac{7}{-5}$

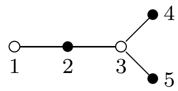
Table 2a: B^{ij} for A_6

(the $\surd$ denotes that the bootstrap is correct and no corrections are needed)

$i\backslash j$	1	2	3	4	5	6
1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
2	–	$(2)_\theta$	$(3)_\theta$	$(4)_\theta$	$(5)_\theta$	$\surd$
3	–	–	$(2)_\theta(4)_\theta$	$(3)_\theta(5)_\theta$	$(4)_\theta$	$\surd$
4	–	–	–	$(2)_\theta(4)_\theta$	$(3)_\theta$	$\surd$
5	–	–	–	–	$(2)_\theta$	$\surd$
6	–	–	–	–	–	$\surd$

Table 2b: Corrections to $v^{ij}(\theta)$ for A_6

$D_5, \qquad h = 8$

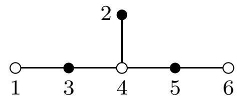


$B^{ij}(\theta)$: $i\backslash j$	1	2	3	4	5
1	$\frac{2\ 8}{0\ -6}$	$\frac{3\ 7}{-1\ -5}$	$\frac{6}{-2}$	$\frac{5}{-3}$	$\frac{5}{-3}$
2	–	$\frac{8}{0}$	$\frac{5\ 7}{-1\ -3}$	$\frac{6}{-2}$	$\frac{6}{-2}$
3	–	–	$\frac{6\ 8}{0\ -2}$	$\frac{7}{-1}$	$\frac{7}{-1}$
4	–	–	–	$\frac{4\ 8}{-2\ -6}$	$\frac{2\ 6}{0\ -4}$
5	–	–	–	–	$\frac{4\ 8}{-2\ -6}$

Table 3a

$i\backslash j$	1	2	3	4	5
1	$\surd$	$\surd$	(4)	$\surd$	$\surd$
2	–	(2)(4)(6)	(3)(5)	(4)	(4)
3	–	–	$(2)(4)^2(6)$	(3)(5)	(3)(5)
4	–	–	–	$\surd$	$\surd$
5	–	–	–	–	$\surd$

Table 3b



$E_6, \qquad h = 12$

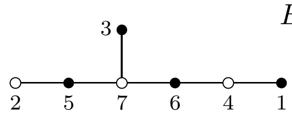
$B^{ij}(\theta)$: $i\backslash j$	1	2	3	4	5	6
1	$\frac{6\ 12}{-4\ -10}$	$\frac{5\ 9}{-3\ -7}$	$\frac{7\ 11}{-3\ -9}$	$\frac{10}{-2}$	$\frac{3\ 9}{-1\ -5}$	$\frac{2\ 8}{0\ -6}$
2	–	$\frac{2\ 8\ 12}{0\ -4\ -10}$	$\frac{10}{-2}$	$\frac{7\ 11}{-1\ -5}$	$\frac{10}{-2}$	$\frac{5\ 9}{-3\ -7}$
3	–	–	$\frac{6\ 12}{-2\ -4}$	$\frac{9\ 11}{-1\ -3}$	$\frac{8\ 10}{0\ -6}$	$\frac{3\ 9}{-1\ -5}$
4	–	–	–	$\frac{8\ 10\ 12}{0\ -2\ -4}$	$\frac{9\ 11}{-1\ -3}$	$\frac{10}{-2}$
5	–	–	–	–	$\frac{6\ 12}{-2\ -4}$	$\frac{7\ 11}{-3\ -9}$
6	–	–	–	–	–	$\frac{6\ 12}{-4\ -10}$

Table 4a

$i\backslash j$	1	2	3	4	5	6
1	$\surd$	$\surd$	(7)	$(4)(6)(8)$	(5)	$\surd$
2	–	(6)	$(4)(6)(8)$	$(3)(5)(7)(9)$	$(4)(6)(8)$	$\surd$
3	–	–	$(2)(4)(6)(8)$	$(3)(5)^2(7)^2(9)$	$(4)(6)(8)(10)$	(5)
4	–	–	–	$(2)(4)^2(6)^3(8)^2(10)$	$(3)(5)^2(7)^2(9)$	$(4)(6)(8)$
5	–	–	–	–	$(2)(4)(6)(8)$	(7)
6	–	–	–	–	–	$\surd$

Table 4b

$E_7, \qquad h = 18$

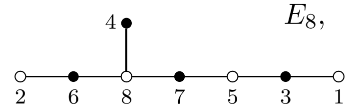


$B^{ij}(\theta)$: $i\backslash j$	1	2	3	4	5	6	7
1	$\frac{2\ 10\ 18}{0\ -8\ -16}$	$\frac{7\ 13}{-5\ -11}$	$\frac{6\ 10\ 14}{-4\ -8\ -12}$	$\frac{3\ 11\ 17}{-1\ -7\ -15}$	$\frac{8\ 14}{-4\ -10}$	$\frac{4\ 12\ 16}{-2\ -6\ -14}$	$\frac{15}{-3}$
2	–	$\frac{2\ 8\ 12\ 18}{0\ -6\ -10\ -16}$	$\frac{5\ 11\ 15}{-3\ -7\ -13}$	$\frac{8\ 14}{-4\ -10}$	$\frac{3\ 13\ 17}{-1\ -5\ -15}$	$\frac{15}{-3}$	$\frac{10\ 16}{-2\ -8}$
3	–	–	$\frac{2\ 14\ 18}{0\ -4\ -16}$	$\frac{15}{-3}$	$\frac{10\ 16}{-2\ -8}$	$\frac{8\ 12\ 16}{-2\ -6\ -10}$	$\frac{13\ 17}{-5\ -1}$
4	–	–	–	$\frac{4\ 10\ 12\ 18}{0\ -6\ -8\ -14}$	$\frac{7\ 13\ 15}{-3\ -5\ -11}$	$\frac{11\ 17}{-1\ -7}$	$\frac{14\ 16}{-2\ -4}$
5	–	–	–	–	$\frac{12\ 18}{0\ -6}$	$\frac{14\ 16}{-2\ -4}$	$\frac{11\ 15\ 17}{-1\ -3\ -7}$
6	–	–	–	–	–	$\frac{10\ 14\ 18}{0\ -4\ -8}$	$\frac{13\ 15\ 17}{-1\ -3\ -5}$
7	–	–	–	–	–	–	$\frac{12\ 14\ 16\ 18}{0\ -2\ -4\ -6}$

Table 5a

$i\backslash j$	1	2	3	4	5	6	7
1	$\surd$	$\surd$	$\surd$	(9)	(6)(12)	(8)(10)	(5)(7)(9)(11)
2	–	$\surd$	(9)	(6)(12)	(7)(9)(11)	(5)(7)(9)(11)(13)	(4)(6)(8)(10)(12)(14)
3	–	–	(6)(8)(10)(12)	(5)(7)(9)(11)(13)	(4)(6)(8)(10)(12)(14)	(4)(6)(8)(10)(12)(14)	$(3)(5)(7)^2(9)^2(11)^2(13)(15)$
4	–	–	–	(2)(8)(10)(16)	(5)(7)(9)(11)(13)	$(3)(5)(7)(9)^2(11)(13)(15)$	$(4)(6)^2(8)^2(10)^2(12)^2(14)$
5	–	–	–	–	$(2)(4)(6)(8)^2(10)^2(12)(14)(16)$	$(4)(6)^2(8)^2(10)^2(12)^2(14)$	$(3)(5)^2(7)^2(9)^3(11)^2(13)^2(15)$
6	–	–	–	–	–	$(2)(4)(6)^2(8)^2(10)^2(12)^2(14)(16)$	$(3)(5)^2(7)^3(9)^3(11)^3(13)^2(15)$
7	–	–	–	–	–	–	$(2)(4)^2(6)^3(8)^4(10)^4(12)^3(14)^2(16)$

Table 5b



$E_8, \qquad h = 30$

$B^{ij}(\theta)$: $i\backslash j$	1	2	3	4	5	6	7	8
1	$\frac{30\ 20\ 12\ 2}{0\ -10\ -18\ -28}$	$\frac{24\ 18\ 14\ 8}{-22\ -16\ -12\ -6}$	$\frac{3\ 13\ 21\ 29}{-1\ -9\ -17\ -27}$	$\frac{7\ 11\ 17\ 21\ 25}{-5\ -9\ -13\ -19\ -23}$	$\frac{4\ 14\ 22\ 28}{-2\ -8\ -16\ -26}$	$\frac{9\ 19\ 25}{-5\ -11\ -21}$	$\frac{5\ 23\ 27}{-3\ -7\ -25}$	$\frac{16\ 26}{-4\ -14}$
2	–	$\frac{2\ 8\ 14\ 20\ 24\ 30}{0 -6\ -10\ -16\ -22\ -28}$	$\frac{9\ 19\ 25}{-5\ -11\ -21}$	$\frac{5\ 23\ 27}{-3\ -7\ -25}$	$\frac{16\ 26}{-4\ -14}$	$\frac{3\ 13\ 19\ 25\ 29}{-1\ -5\ -11\ -17\ -27}$	$\frac{11\ 17\ 21\ 27}{-3\ -9\ -13\ -19}$	$\frac{22\ 28}{-2\ -8}$
3	–	–	$\frac{4\ 12\ 14\ 20\ 22\ 30}{0\ -8\ -10\ -16\ -18\ -26}$	$\frac{16\ 26}{-4\ -14}$	$\frac{5\ 13\ 21\ 23\ 29}{-1\ -7\ -9\ -17\ -25}$	$\frac{8\ 18\ 24\ 26}{-4\ -6\ -12\ -22}$	$\frac{22\ 28}{-2\ -8}$	$\frac{17\ 25\ 27}{-3\ -5\ -13}$
4	–	–	–	$\frac{2\ 12\ 16\ 20\ 26\ 30}{0\ -4\ -10\ -14\ -18\ -28}$	$\frac{9\ 19\ 23\ 27}{-3\ -7\ -11\ -21}$	$\frac{22\ 28}{-2\ -8}$	$\frac{14\ 18\ 24\ 28}{-2\ -6\ -12\ -16}$	$\frac{21\ 25\ 29}{-1\ -5\ -9}$
5	–	–	–	–	$\frac{12\ 20\ 22\ 30}{0\ -8\ -10\ -18}$	$\frac{17\ 25\ 27}{-3\ -5\ -13}$	$\frac{21\ 25\ 29}{-1\ -5\ -9}$	$\frac{18\ 24\ 26\ 28}{-2\ -4\ -6\ -12}$
6	–	–	–	–	–	$\frac{14\ 20\ 24\ 30}{0\ -6\ -10\ -16}$	$\frac{16\ 22\ 26\ 28}{-2\ -4\ -8\ -14}$	$\frac{21\ 23\ 27\ 29}{-1\ -3\ -7\ -9}$
7	–	–	–	–	–	–	$\frac{20\ 24\ 26\ 30}{0\ -4\ -6\ -10}$	$\frac{19\ 23\ 25\ 27\ 29}{-1\ -3\ -5\ -7\ -11}$
8	–	–	–	–	–	–	–	$\frac{20\ 22\ 24\ 26\ 28\ 30}{0\ -2\ -4\ -6\ -8\ -10}$

Table 6a

$i\backslash j$	1	2	3	4	5	6	7	8
1	$\surd$	$\surd$	11	(15)	10 12	7 13 (15)	9 11 13 (15)	6 8 10 12 14
2	–	12	7 13 (15)	9 11 13 (15)	6 8 10 12 14	7 9 11 13 (15)	5 7 9 11 13 $(15)^2$	4 6 8 10^2 12^2 14^2
3	–	–	2 10 12	6 8 10 12 14	3 9 11^2 13 (15)	6 8 10 12 14^2	4 6 8 10^2 12^2 14^2	5 7^2 9^2 11^2 13^2 $(15)^3$
4	–	–	–	6 8 10 12 14	5 7 9 11 13^2 $(15)^2$	4 6 8 10^2 12^2 14^2	4 6 8^2 10^2 12^2 14^2	3 5 7^2 9^2 11^3 13^3 $(15)^3$
5	–	–	–	–	2 4 6 8 10^2 12^2 14^2	5 7^2 9^2 11^2 13^2 $(15)^3$	3 5 7^2 9^2 11^3 13^3 $(15)^3$	4 6^2 8^3 10^3 12^3 14^4
6	–	–	–	–	–	2 4 6 8^2 10^2 12^3 14^2	4 6^2 8^2 10^3 12^3 14^3	3 5^2 7^2 9^3 11^4 13^4 $(15)^4$
7	–	–	–	–	–	–	2 4 6^2 8^3 10^3 12^4 14^4	3 5^2 7^3 9^4 11^4 13^5 $(15)^5$
8	–	–	–	–	–	–	–	2 4^2 6^3 8^4 10^5 12^6 14^6

Table 6b: with $x = (x)(h - x)$

The soliton bootstrap completed: the bootstrap equation for $\tilde{v}^{ij}(\theta)$

We remark that with $\tilde{v}^{ij}(\theta)$ defined through equation (12.2) and $v^{ij}(\theta) = B^{ij}(\theta)^{-1}\tilde{v}^{ij}(\theta)$, the bootstrap equation for $\tilde{v}^{ij}(\theta)$ follows from the classical bootstrap equation (3.10), except for possibly adjusting the arguments of $S_{\tilde{q}^{-h}}(w)$ by $e^{2\pi i}$. This adjustment introduces factors of the form $(1-x^hq^p)^{-1}$ into the bootstrap equation, which have already been discussed for the A_n case, in section 11. For these A_n theories, the extra factors fit in precisely with the factors which we can compute when we fuse the R-matrices. For A_{h-1}, from (11.4), we have

$$\begin{aligned} R^{11}(x) &= (1-x^hq^2)P_0 + (x^h - q^2)P_2 \\ R^{12}(x) &= (1-x^hq^3)P_{0'} + (x^h - q^3)P_3, \end{aligned} \tag{13.9}$$

where P_2 and P_3 project onto the fundamental representations V_2 and V_3 respectively. Both R^{11} and R^{12} are of order one. However, if we fuse $R^{11}(x)$, using the fusing $1+1\to 2$,

$$g(x)R^{12}(x) = (P_2\otimes 1)(1\otimes R^{11}(xq^{-1/h}))(R^{11}(xq^{1/h})\otimes 1)(1\otimes P_2), \tag{13.10}$$

by comparing orders, we see that $g(x)$ must be a non-trivial factor with order one. Now $R^{11}(1) = (1-q^2)1$, $R^{11}(q^{2/h}) = (1-q^4)P_0$, and $P_2P_0 = 0$, so that if we set $x = q^{1/h}$ in (13.10), then the $(P_2\otimes 1)$ in the left most position acts on the $(R^{11}(q^{2/h})\otimes 1)$ giving zero, so that $g(q^{1/h}) = 0$, and we infer that $g(x) = (1-x^hq^{-1})$. Note that this agrees with (9.32), for $su(3)$ or $h=2$.

For A_6, we note that, from (12.2)

$$\begin{aligned} \tilde{v}^{11}(\theta) &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-i\pi})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{i\pi})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{5\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{5\pi i}{7}})} \\ \tilde{v}^{12}(\theta) &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{6\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{6\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{4\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{4\pi i}{7}})}, \end{aligned} \tag{13.11}$$

and we check $\tilde{v}^{11}(\theta+\frac{i\pi}{7})\tilde{v}^{11}(\theta-\frac{i\pi}{7}) = f(x)\tilde{v}^{12}(\theta)$

$$\begin{aligned} &\tilde{v}^{11}(\theta+\frac{i\pi}{7})\tilde{v}^{11}(\theta-\frac{i\pi}{7}) \\ &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{8\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{6\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{6\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{4\pi i}{7}})}\frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{6\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{8\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{4\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{6\pi i}{7}})} \\ &= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{6\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{8\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{4\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{4\pi i}{7}})} \\ &= \tilde{v}^{12}(\theta)\frac{1}{1-x^hq^{-1}}. \end{aligned} \tag{13.12}$$

Now consider a case where we have to use the corrections in Table 1. For example,

$$\tilde{v}^{32}(\theta) = \frac{S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{\frac{6\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{6\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^he^{-2\pi i+\theta}e^{\frac{2\pi i}{7}})S_{\tilde{q}^{-h}}(\tilde{q}^he^\theta e^{-\frac{2\pi i}{7}})}$$

$$\tilde{v}^{42}(\theta) = \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\frac{5\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{5\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{\pi i}{7}})}, \tag{13.13}$$

and for the fusing $1+3 \to 4$, we check $\tilde{v}^{32}(\theta + \frac{\pi i}{7})\tilde{v}^{12}(\theta - \frac{3\pi i}{7}) = f(x)\tilde{v}^{42}(\theta)$

$$\begin{aligned}
&\tilde{v}^{32}(\theta + \frac{\pi i}{7})\tilde{v}^{12}(\theta - \frac{3\pi i}{7}) \\
&= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\pi i}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{5\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{3\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{\pi i}{7}})} \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{\frac{3\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{9\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\pi i})} \\
&= \frac{S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{5\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{9\pi i}{7}})}{S_{\tilde{q}^{-h}}(\tilde{q}^h e^{-2\pi i+\theta} e^{\frac{\pi i}{7}}) S_{\tilde{q}^{-h}}(\tilde{q}^h e^\theta e^{-\frac{\pi i}{7}})} \frac{1}{1-x^h} \frac{1}{1-x^h q^4} \\
&= \tilde{v}^{42}(\theta) \frac{(1-x^h q^2)}{(1-x^h)(1-x^h q^4)}.
\end{aligned} \tag{13.14}$$

For the fusion

$$g(x)R^{42}(x) = (I \otimes R^{32}(xq^{-1/7}))(R^{12}(xq^{3/7}) \otimes 1)\Big|_{V_4 \otimes V_2}, \tag{13.15}$$

Hollowood gives a prescription [3] for the zeroes of $R^{42}(x)$, defined by setting $g(x) = 1$, throughout the fusion procedure. This is found by repeatedly applying the argument (13.10).

It is argued that the zeroes of $R^{ab}(x)$, for $b \geq a$, are at

$$x = q^{-\frac{1}{h}(a+b-2j-2k)}, \qquad j = 1, 2, \ldots, a, \quad k = 1, 2, \ldots, b-1,$$

so the four zeroes of $R^{32}(x)$ are at

$$x = q^{-\frac{1}{h}}, \quad q^{\frac{1}{h}}, \quad q^{\frac{1}{h}}, \quad q^{\frac{3}{h}}.$$

The zero of $R^{12}(x)$ is at $x = q^{\frac{1}{h}}$, and the zeroes of $R^{42}(x)$ are at

$$x = q^{-\frac{2}{h}}, \quad 1, \quad 1, \quad q^{\frac{2}{h}}, \quad q^{\frac{2}{h}}, \quad q^{\frac{4}{h}}.$$

In the fusion (13.15), the zeroes of $R^{32}(x)$ and $R^{12}(x)$ cancel with five out of the siz zeroes of $R^{42}(x)$, leaving a single zero at $x = 1$, hence we infer that $g(x) = (1 - x^h)$. This accounts for the factor $(1 - x^h)^{-1}$ on the right-hand side of (13.14). The factor $(1 - x^h q^4)^{-1}$ is accounted for, since it is the 42 correction from Table 1, and $(1 - x^h q^2)$ is the 32 correction taken to the other side of the bootstrap equation.

In a similar fashion, it is possible to show for the outstanding A_n cases that the normalisations arising from fusion agree with the factors thrown up in the bootstrap on $\tilde{v}^{ij}$, and the factors in Table 1.

14 The lowest mass breather bootstrap, and comparison with the Toda particle S-matrices

We follow through the bootstrap for the lowest mass breathers and calculate the lowest mass breather – lowest mass breather S-matrices $S^{b_i b_j}(\theta)$. Recall the notation for the building blocks, from [11],

$$(x)_\theta = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})},$$

and

$$\{x\}_\theta = \frac{(x-1)_\theta (x+1)_\theta}{(x-1+B)_\theta (x+1-B)_\theta},$$

where

$$B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \frac{\beta^2}{4\pi}}, \tag{14.1}$$

the β in (14.1) is real for the affine Toda particles, and purely imaginary for the solitons.

Recall also the general formula for the Toda particle S-matrices [12],

$$S^{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_\theta^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}}. \tag{14.2}$$

The lowest mass breather pole in $S^{i\bar{i}}_{\lambda\bar{\lambda}}(\theta)$ is at

$$\theta = i\pi - \frac{i\gamma}{2h}.$$

We always take the transmissive soliton S-matrix elements, rather than the reflective, since only then does the bootstrap make sense.

$$\begin{aligned} S^{s_i b_j}_{\lambda}(\theta) &= S^{ij}_{\lambda\mu}\Big(\theta - i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) S^{i\bar{j}}_{\lambda\bar{\mu}}\Big(\theta + i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) \\ &= S^{ij}_{\lambda\mu}\Big(\theta - i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) S^{ij}_{\lambda\mu}\Big(-\theta + i\Big(\frac{\pi}{2} + \frac{\gamma}{4h}\Big)\Big) \end{aligned} \tag{14.3}$$

$$\begin{aligned} S^{\bar{s}_i b_j}_{\lambda}(\theta) &= S^{\bar{i}j}_{\bar{\lambda}\mu}\Big(\theta - i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) S^{\bar{i}\bar{j}}_{\bar{\lambda}\bar{\mu}}\Big(\theta + i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) \\ &= S^{ij}_{\lambda\mu}\Big(-\theta + i\pi + i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big) S^{ij}_{\lambda\mu}\Big(\theta + i\Big(\frac{\pi}{2} - \frac{\gamma}{4h}\Big)\Big). \end{aligned} \tag{14.4}$$

We will ignore all the polynomials in x^h of the form $\prod_p (1 - x^h q^p)$.

With this proviso in mind, using the formula (10.8), so we also ignore the corrections of the form $(p)_\theta$, we have

$$S^{s_i b_j}_{\lambda}(\theta) = \frac{X^{ij}_q(\theta - i(\frac{\pi}{2} - \frac{\gamma}{4h}))}{X^{ij}_q(-\theta + i(\frac{\pi}{2} - \frac{\gamma}{4h}))} \frac{X^{ij}_q(-\theta + i(\frac{\pi}{2} + \frac{\gamma}{4h}))}{X^{ij}_q(\theta - i(\frac{\pi}{2} + \frac{\gamma}{4h}))} \tag{14.5}$$

$$S^{\bar{s}_i b_j}_{\bar{\lambda}}(\theta) = \frac{X^{ij}_q(-\theta+i\pi+i(\frac{\pi}{2}-\frac{\gamma}{4h}))}{X^{ij}_q(\theta-i\pi-i(\frac{\pi}{2}-\frac{\gamma}{4h}))}\frac{X^{ij}_q(\theta+i(\frac{\pi}{2}-\frac{\gamma}{4h}))}{X^{ij}_q(-\theta-i(\frac{\pi}{2}-\frac{\gamma}{4h}))}$$

$$= \frac{1}{\prod_p(1-x^{-h}q^p)\prod_r(1-x^h q^r)}\frac{X^{ij}_q(-\theta-i(\frac{\pi}{2}+\frac{\gamma}{4h}))}{X^{ij}_q(\theta+i(\frac{\pi}{2}+\frac{\gamma}{4h}))}\frac{X^{ij}_q(\theta+i(\frac{\pi}{2}-\frac{\gamma}{4h}))}{X^{ij}_q(-\theta-i(\frac{\pi}{2}-\frac{\gamma}{4h}))}$$

We see, with the above proviso again in mind, that

$$S^{\bar{s}_i b_j}_{\bar{\lambda}}(\theta) = S^{s_i b_j}_{\lambda}(-\theta)^{-1}. \tag{14.6}$$

Then

$$S^{s_i b_j}_{\lambda}(\theta) = \frac{X^{ij}_q(\theta-i(\frac{\pi}{2}-\frac{\gamma}{4h}))}{X^{ij}_q(\theta-i(\frac{\pi}{2}+\frac{\gamma}{4h}))}\frac{X^{ij}_q(-\theta+i(\frac{\pi}{2}+\frac{\gamma}{4h}))}{X^{ij}_q(-\theta+i(\frac{\pi}{2}-\frac{\gamma}{4h}))}$$

can be written as a product of sinh's:
with

$$X^{ij}_q(\theta) = \prod_{p=0}^{h-1} S_{\tilde{q}^{-h}}\left(e^{\frac{isp\gamma}{4h}}e^{\theta}e^{\frac{i\pi}{2h}(c(i)-c(j))}e^{\frac{2\pi ip}{h}-i\pi}\right)^{\frac{\gamma_i\cdot\sigma^p\gamma_j}{2}},$$

$$S^{s_i b_j}_{\lambda}\left(\theta+i\left(\frac{\pi}{2}-\frac{\gamma}{4h}\right)\right) =$$

$$\prod_{p=0}^{h-1}\sinh\left(\frac{\theta}{2}+\frac{\pi ip}{h}+\frac{i\pi}{4h}(c(i)-c(j))+\frac{i\gamma}{8h}(s_p-1)\right)^{-\frac{\gamma_i\cdot\sigma^p\gamma_j}{2}}$$

$$\times\prod_{p=0}^{h-1}\sinh\left(-\frac{\theta}{2}+\frac{\pi ip}{h}+\frac{i\pi}{4h}(c(i)-c(j))+\frac{i\gamma}{8h}(s_p+1)\right)^{-\frac{\gamma_i\cdot\sigma^p\gamma_j}{2}}. \tag{14.7}$$

We now use [15]

$$\gamma_i\cdot\sigma^p\gamma_j = \lambda_i\cdot\sigma^p(1-\sigma)\sigma^{\frac{c(i)-1}{2}}\gamma_j, \tag{14.8}$$

and then,

$$S^{s_i b_j}_{\lambda}\left(\theta+i\left(\frac{\pi}{2}-\frac{\gamma}{4h}\right)\right) =$$

$$\frac{\prod_{q=1}^{h}\sinh\left(\frac{\theta}{2}+\frac{\pi iq}{h}-\frac{i\pi}{4h}(c(i)+c(j))+\frac{\pi i}{2h}+\frac{i\gamma}{8h}(s_{q-\frac{c(i)-1}{2}}-1)\right)^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}}}{\prod_{q=1}^{h}\sinh\left(\frac{\theta}{2}+\frac{\pi iq}{h}-\frac{i\pi}{4h}(c(i)+c(j))-\frac{\pi i}{2h}+\frac{i\gamma}{8h}(s_{q-\frac{c(i)+1}{2}}-1)\right)^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}}}$$

$$\times\frac{\prod_{q=1}^{h}\sinh\left(-\frac{\theta}{2}+\frac{\pi iq}{h}-\frac{i\pi}{4h}(c(i)+c(j))+\frac{\pi i}{2h}+\frac{i\gamma}{8h}(s_{q-\frac{c(i)-1}{2}}+1)\right)^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}}}{\prod_{q=1}^{h}\sinh\left(-\frac{\theta}{2}+\frac{\pi iq}{h}-\frac{i\pi}{4h}(c(i)+c(j))-\frac{\pi i}{2h}+\frac{i\gamma}{8h}(s_{q-\frac{c(i)+1}{2}}+1)\right)^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}}}.$$

Now

$$\begin{aligned} S^{b_i b_j}(\theta) &= S_{\lambda}^{s_i b_j}\left(\theta + i\left(\frac{\pi}{2} - \frac{\gamma}{4h}\right)\right) S_{\bar{\lambda}}^{\bar{s}_i b_j}\left(\theta - i\left(\frac{\pi}{2} - \frac{\gamma}{4h}\right)\right) \\ &= \frac{S_{\lambda}^{s_i b_j}(\theta + i(\frac{\pi}{2} - \frac{\gamma}{4h}))}{S_{\lambda}^{s_i b_j}(-\theta + i(\frac{\pi}{2} - \frac{\gamma}{4h}))}. \end{aligned} \tag{14.9}$$

We observe that

$$S^{b_i b_j}(\theta) = \frac{S(\theta)}{S(-\theta)},$$

where

$$S(\theta) = \prod_{q=1}^{h} \left(\frac{\sinh(\frac{\theta}{2} + \frac{\pi i q}{h} - \frac{i\pi}{4h}(c(i) + c(j)) + \frac{\pi i}{2h} + \frac{i\gamma}{8h}(s_{q-\frac{c(i)-1}{2}} - 1))}{\sinh(\frac{\theta}{2} + \frac{\pi i q}{h} - \frac{i\pi}{4h}(c(i) + c(j)) - \frac{\pi i}{2h} + \frac{i\gamma}{8h}(s_{q-\frac{c(i)+1}{2}} - 1))} \right.$$
$$\left. \times \frac{\sinh(\frac{\theta}{2} + \frac{\pi i q}{h} - \frac{i\pi}{4h}(c(i) + c(j)) - \frac{\pi i}{2h} + \frac{i\gamma}{8h}(s_{q-\frac{c(i)+1}{2}} + 1))}{\sinh(\frac{\theta}{2} + \frac{\pi i q}{h} - \frac{i\pi}{4h}(c(i) + c(j)) + \frac{\pi i}{2h} + \frac{i\gamma}{8h}(s_{q-\frac{c(i)-1}{2}} + 1))} \right)^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}}. \tag{14.10}$$

We have obtained $S(\theta)$ from $S_{\lambda}^{s_i b_j}(\theta + i(\frac{\pi}{2} - \frac{\gamma}{4h}))$, by exchanging the factors of the form $\sinh(-\theta/2 + A)$ in the second half of $S_{\lambda}^{s_i b_j}(\theta + i(\frac{\pi}{2} - \frac{\gamma}{4h}))$ with $\sinh(\theta/2 + A)^{-1}$. We write this as

$$S^{b_i b_j}(\theta) = \prod_{q=1}^{h} \left[2q - \frac{c(i) + c(j)}{2} \right]^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}}, \tag{14.11}$$

with $\left[2q - \frac{c(i)+c(j)}{2}\right]$ defined in the obvious way as

$$\left(\frac{(2q - \frac{c(i)+c(j)}{2} + 1 + \frac{\gamma}{4\pi}(s_{q-\frac{c(i)-1}{2}} - 1))_\theta \, (2q - \frac{c(i)+c(j)}{2} - 1 + \frac{\gamma}{4\pi}(s_{q-\frac{c(i)+1}{2}} + 1))_\theta}{(2q - \frac{c(i)+c(j)}{2} - 1 + \frac{\gamma}{4\pi}(s_{q-\frac{c(i)+1}{2}} - 1))_\theta \, (2q - \frac{c(i)+c(j)}{2} + 1 + \frac{\gamma}{4\pi}(s_{q-\frac{c(i)-1}{2}} + 1))_\theta} \right). \tag{14.12}$$

Suppose that

$$s_{q-\frac{c(i)+1}{2}} = -1, \qquad s_{q-\frac{c(i)-1}{2}} = 1, \tag{14.13}$$

for $\lambda_i \cdot \sigma^q \gamma_j \neq 0$, then that value of q contributes the block

$$\left[2q - \frac{c(i) + c(j)}{2} \right]^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}} = \left(\frac{(2q - \frac{c(i)+c(j)}{2} + 1)_\theta (2q - \frac{c(i)+c(j)}{2} - 1)_\theta}{(2q - \frac{c(i)+c(j)}{2} + 1 + \frac{\gamma}{2\pi})_\theta (2q - \frac{c(i)+c(j)}{2} - 1 - \frac{\gamma}{2\pi})_\theta} \right)^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}}$$

to $S^{b_i b_j}(\theta)$. We see how the correct block structure of $\{2q - \frac{c(i)+c(j)}{2}\}_\theta$ has emerged, and this factor becomes

$$\left[2q - \frac{c(i) + c(j)}{2} \right]^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}} = \left\{ 2q - \frac{c(i) + c(j)}{2} \right\}_\theta^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}},$$

after noting that

$$B(\beta) = \frac{1}{2\pi}\frac{\beta^2}{1+\frac{\beta^2}{4\pi}} = -\frac{\gamma}{2\pi}.$$

The case (14.13) can not occur for all values of q, since there are cases when $\lambda_i \cdot \sigma^q \gamma_j \neq 0$, and $\lambda_i \cdot \sigma^{q+1}\gamma_j \neq 0$, i.e. consecutive blocks.

However, for the case of the fundamental element $S^{11}(\theta)$, there are consecutive blocks with $\lambda_1 \cdot \sigma^q \gamma_1 = \pm 1$, and $\lambda_1 \cdot \sigma^{q+1}\gamma_1 = \mp 1$, which occur when $\gamma_1 \cdot \sigma^p \gamma_1 = \pm 2$, that is $p = 0$, or $p = \frac{h}{2}$ (if $\bar{1} = 1$), respectively. It is important to note that all the remaining blocks are non-consecutive, this can be shown by inspecting the data $\lambda_1 \cdot \sigma^q \gamma_1$, $q = 1, \ldots, h$, on a case by case basis. This data is available, for example, in [11], [12], or [8].

Suppose that the block $\lambda_1 \cdot \sigma^q \gamma_1 \neq 0$, and that it is isolated. Suppose also that q is in the second half of its range, noting that $\lambda_i \cdot \sigma^q \gamma_j = -\lambda_i \cdot \sigma^{q'}\gamma_j$, with $q' = h - q + \frac{c(i)+c(j)}{2}$. It suffices to show the result in this range. Also in this range $\lambda_1 \cdot \sigma^q \gamma_1 > 0$. Then, the only way for this to occur, from the recursion formula (14.8), is if

$$\gamma_1 \cdot \sigma^{q-\frac{c(i)+1}{2}}\gamma_1 = -1, \qquad \text{and} \qquad \gamma_1 \cdot \sigma^{q-\frac{c(i)-1}{2}}\gamma_1 = 1,$$

with $\lambda_1 \cdot \sigma^q \gamma_1 = 1$. In this range

$$s_{q-\frac{c(i)+1}{2}} = -1, \qquad \text{and} \qquad s_{q-\frac{c(i)-1}{2}} = 1,$$

as required to form a correct block at q.

Now consider the situation when $\gamma_1 \cdot \sigma^p \gamma_1 = 2$, at $p = 0$, and suppose that $c(1) = 1$. There are two consecutive blocks here, but these two are isolated. From (14.8)

$$\lambda_1 \cdot \sigma^{h-1}\gamma_1 = 0, \quad \lambda_1 \cdot \sigma^h \gamma_1 = 1 \quad \lambda_1 \cdot \sigma^1 \gamma_1 = -1 \quad \lambda_1 \cdot \sigma^2 \gamma_1 = 0,$$

it also follows that (since the two blocks are isolated)

$$s_{h-1} = -1, \quad s_0 = 1, \quad s_1 = 1.$$

The block at $q = h$ is evidently correctly formed, since it satisfies the condition (14.13). The block at $q = 1$ is, from equation (14.12),

$$\Big[2q-1\Big]_{q=1}^{-\frac{\lambda_i \cdot \sigma^q \gamma_j}{2}} = \left(\frac{(1+1)_\theta (1-1+\frac{\gamma}{2\pi})_\theta}{(1-1)_\theta (1+1+\frac{\gamma}{2\pi})_\theta}\right)^{\frac{1}{2}},$$

but $(x)_\theta = (-x)_\theta^{-1}$, and $(0)_\theta = 1$, so

$$= \left(\frac{(1+1)_\theta (1-1)_\theta}{(1-1-\frac{\gamma}{2\pi})_\theta (1+1+\frac{\gamma}{2\pi})_\theta}\right)^{\frac{1}{2}} = \{1\}_\theta^{\frac{1}{2}}.$$

Now consider the situation when $\gamma_1 \cdot \sigma^p \gamma_1 = -2$, at $p = \frac{h}{2} = H$, if $\bar{1} = 1$, and suppose that $c(1) = 1$. Then

$$\lambda_1 \cdot \sigma^{H-1}\gamma_1 = 0, \quad \lambda_1 \cdot \sigma^H \gamma_1 = -1 \quad \lambda_1 \cdot \sigma^{H+1}\gamma_1 = 1 \quad \lambda_1 \cdot \sigma^{H+2}\gamma_1 = 0,$$

and

$$s_{H-1}=-1, \quad s_H=1, \quad s_{H+1}=1.$$

The block at $q=H$ is correctly formed, and the block at $q=H+1$ is

$$\Big[2q-1\Big]_{q=H+1}^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}} = \left(\frac{(2H+1+1)_\theta(2H+1-1+\frac{\gamma}{2\pi})_\theta}{(2H+1-1)_\theta(2H+1+1+\frac{\gamma}{2\pi})_\theta}\right)^{-\frac{1}{2}}.$$

Now $(x)_\theta=(2h-x)_\theta^{-1}$, so

$$(2H+1-1+\frac{\gamma}{2\pi})_\theta=(h-\frac{\gamma}{2\pi})_\theta^{-1}=(2H+1-1-\frac{\gamma}{2\pi})_\theta^{-1},$$

and $(2H+1-1)_\theta=(2H+1-1)_\theta^{-1}$. Hence

$$\Big[2q-1\Big]_{q=H+1}^{-\frac{\lambda_i\cdot\sigma^q\gamma_j}{2}} = \left(\frac{(2H+1+1)_\theta(2H+1-1)_\theta}{(2H+1-1-\frac{\gamma}{2\pi})_\theta(2H+1+1+\frac{\gamma}{2\pi})_\theta}\right)^{-\frac{1}{2}},$$

as required. The case $c(1)=-1$ is left to the reader.

This shows that we get the same as the particle S-matrix, in the case of the fundamental element, $i=j=1$. For the remaining elements, we merely remark that we can run through the bootstrap on the soliton result $v^{ij}(\theta)$, and that this is directly equivalent to running through the bootstrap on the lowest mass breather S-matrix starting from $S^{b_1b_1}$, since the fusing angles for the lowest mass breathers and solitons are precisely the same. The agreement of the result with the particle S-matrix for the fundamental element, guarantees that it will agree for all i, and j.

We have previously run through the soliton bootstrap on the ground state solitons, and obtained the corrections of the form $(p)_\theta$. These corrections are actually precisely the corrections needed to turn the incorrectly formed blocks, which occured when they were consecutive in (14.12), into correctly formed ones.

15 Discussion and conclusions

In this paper, we have given exact expressions for the soliton S-matrices in the simply-laced affine Toda field theories, found by 'q-deforming' the classical time delays, by inserting the regularised quantum dilogarithms. We have then checked crossing and unitarity, which essentially follow from analogous properties satisfied by the time delays. The time delays also satisfy the bootstrap equation, in these Toda theories, which allows the bootstrap property of the exact S-matrix to be partially checked. However, in checking the bootstrap, we throw up two types of factor, one of type $(p)_\theta$, and the other of type $(1-x^hq^p)^{-1}$, and the expressions found for the S-matrix have to be corrected by multiplying by factors of both these types. These additional corrections trivially satisfy crossing and unitarity, by themselves, except for the unitarity of the corrections $(1-x^hq^p)^{-1}$. These are precisely required in order to satisfy the unitarity condition. The corrections introduce poles onto the physical strip which either enhance poles already present, due to fusing solitons, or they are new. The new ones should have an interpretation in terms of Landau singularities.

For the A_n theories, it is possible to determine all the corrections, since we have exact R-matrices intertwining between tensor products of all the fundamental representations, and in

addition it is possible to determine the precise normalisation (or zeroes) of fused R-matrices. For the A_n case, the bootstrap property is fully checked in this paper. However the paper is incomplete, since the corrections of type $(1 - x^h q^p)^{-1}$ are not determined for the D and E series. This is because the fusion procedure for the R-matrices is much more subtle in these cases, and the precise normalisation or zeroes of fused R-matrices has not yet been determined. This means that the bootstrap has only been partially checked for these cases. A crude way of determining them would be to fuse the R-matrices explicitly using a computer algebra package. This could be difficult, however, because of the sizes of the matrices involved.

It is also shown in this paper, that the S-matrix of the lowest mass breathers $S^{b_i b_j}(\theta)$ is the same as the particle S-matrix $S^{ij}(\theta)$ known previously in the affine Toda field theories. In this calculation, we have ignored the factors of the type $(1 - x^h q^p)^{-1}$, these include the individual elements of the R-matrix. This is an important independent check on the solutions. The excited solitons are not discussed much, beyond pointing out that they exist. It would be interesting to try to show full closure of the bootstrap on all these excited states.

The reader may also have noticed that quantum groups and R-matrices have not been discussed in any great detail. In particular, we have avoided all discussion of any deeper relation between affine quantum groups and the symmetries of quantum integrable models. The author believes that such a relation may indeed be true, but since the S-matrix must satisfy the quantum Yang-Baxter equation, the results presented here are valid, even if the relation is not true, provided the missing charge problem can be solved. It will be the subject of future publications to study this relation, and to see how the quantum dilogarithms appear naturally in a quantisation of the models. Recall that, classically, the time delays, $X^{ij}(\theta)$, appeared quite naturally from the vertex operator representations of the affine Kac-Moody algebras.

Acknowledgements

Prof. David Olive brought the regularised quantum dilogarithm to my attention, after seeing it at Faddeev's 60$^{\text{th}}$ birthday celebration. Thanks go to Tim Hollowood for pointing out how the zeroes which occur in the bootstrap can be computed in the A_n case. Thanks must also go to Edwin Beggs for encouragement.

The author was mainly supported by EPSRC, in the form of a studentship under which most of this work was carried out. Further work was conducted, and the paper written, whilst the author was supported by PPARC.

References

[1] P.E. Dorey, 'Root systems and purely elastic S-matrices I', *Nucl. Phys.* **B358** (1991) 654.

[2] T.J. Hollowood, 'Quantising $sl(N)$ solitons and the Hecke algebra', *Int. J. Mod. Phys.* **A8** (1993), 947-980

[3] T.J. Hollowood, 'The analytic structure of trigonometric S-matrices', *Nucl. Phys.* **B414** (1994) 379-404.

[4] G.W. Delius, 'Exact S-matrices with affine quantum group symmetry', *Nucl. Phys.* **B451** (1995) 445-468

[5] A.B. Zamolodchikov, 'Exact two-particle S-matrix of quantum solitons of the sine-Gordon model', *JETP Lett.* **25**, No. 10, (1977) 468-471;
A.B. Zamolodchikov and Al. B. Zamolodchikov, 'Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models', *Ann. Phys.* **120** (1979) 253.

[6] M. Karowski, H.J. Thun, T.T. Truong and P.H. Weisz, 'On the uniqueness of a purely elastic S-matrix in (1+1) dimensions', *Phys. Lett.* **67B** (1977) 321-322.

[7] L.D. Faddeev, 'Current-like variables in massive and massless integrable models', lectures delivered at the international school of physics 'Enrico Fermi' 28 June-8 July, **hep-th 9408041**

[8] P.R. Johnson, 'On solitons in affine Toda field theories', *PhD. Thesis*, University of Wales (1996)

[9] A.Fring, P.R. Johnson, M.A.C. Kneipp, D.I. Olive, 'Vertex operators and soliton time delays in affine Toda field theory', *Nucl. Phys.* **B430** (1994) 597-614

[10] R. Jackiw and G. Woo, 'Semiclassical scattering of quantized non-linear waves', *Phys. Rev.* **D12** (1975) 1643-1649

[11] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, 'Affine Toda field theory and exact S-matrices', *Nucl. Phys.* **B338** (1990) 689-746

[12] P.E. Dorey, 'Root systems and purely elastic S-matrices II', *Nucl. Phys.* **B374** (1992) 741-761.

[13] L.D. Faddeev and V.E. Korepin, 'Quantum theory of solitons', *Phys. Rep.* **42** (1978) 1-87.

[14] G.M. Gandenberger, 'Exact S-matrices for bound states of A_2^1 affine Toda solitons', *Nucl. Phys.* **B449** (1995) 375-405

[15] A.Fring and D.I.Olive, 'The fusing rule and the scattering matrix of affine Toda theory', *Nucl. Phys.* **B379** (1992) 429-447

[16] T.J. Hollowood, 'Solitons in affine Toda field theories', *Nucl. Phys.* **B384** (1992) 523-540

[17] E.J. Beggs and P.R. Johnson, 'Inverse scattering and solitons in A_{n-1} affine Toda field theories', University of Wales Swansea, Dept. of Physics, preprint **SWAT 95-96/115**

[18] W.A. McGhee, 'The topological charges of A_n affine Toda solitons', *Int. J. Mod. Phys.* **A9**, (1994) 2645-2666

[19] D.I. Olive, N. Turok and J.W.R. Underwood, 'Solitons and the energy-momentum tensor for affine Toda theory', *Nucl. Phys.* **B401** (1993) 663-697.

[20] D.I. Olive, N. Turok and J.W.R. Underwood, 'Affine Toda solitons and vertex operators', *Nucl. Phys.* **B409** (1993) 509-546.

[21] S. Coleman, 'Semi-classical crossing symmetry for semi-classical soliton scattering', *Phys. Rev.* **D12** (1975) 1650-1651.

[22] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, 'The analytic S-matrix', Cambridge University Press (1966)

[23] T.J. Hollowood, 'Quantum soliton mass corrections in SL(N) affine Toda field theory', *Phys. Lett.* **B300** (1993) 73-83

[24] N.J. MacKay and G.M.T. Watts, 'Quantum mass corrections for affine Toda solitons', *Nucl. Phys.* **B441** (1995) 277-309

[25] M. Jimbo, 'A q-difference analog of $U(G)$ and the Yang-Baxter equation', *Lett. Math. Phys.* **10** (1985) 63-69;
M. Jimbo, 'A q-analog of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation', *Lett. Math. Phys.* **11** (1986) 247.

[26] D.I. Olive and N. Turok, 'Algebraic structure of Toda systems', *Nucl. Phys.* **B220** (1983) 491-507

[27] D.I. Olive and N. Turok, 'The Toda lattice field theory heirarchies and zero-curvature conditions in Kac-Moody algebras', *Nucl. Phys.* **B265** (1986) 469-484

[28] E. Ogievetsky, N. Reshetikhin, P. Wiegmann, 'The principal chiral field in two dimensions on classical Lie algebras: The Bethe-ansatz solution and factorized theory of scattering', *Nucl. Phys.* **B280** (1987) 45-96.
E. Ogievetsky and P. Wiegmann, 'Factorised S-matrix and the Bethe ansatz for simple Lie groups', *Phys. Letts.* **B168** (1986) 360-366

[29] L.D. Faddeev and R.M. Kashaev, 'Quantum dilogarithm', *Mod. Phys. Lett.* **A9** (1994) 427-434

[30] V.E. Korepin, P.P. Kulish, and L.D. Faddeev, 'Soliton quantization', *JETP Lett.* **21**,No. 5, (1975) 138-139.

[31] M. Karowski and H.J. Thun, 'Complete S-matrix of the massive Thirring model', *Nucl. Phys.* **B130** (1977) 295-308.

[32] G.W. Delius, M.D. Gould and Y-Z. Zhang, 'On the construction of trigonometric solutions of the Yang-Baxter equation', *Nucl. Phys.* **B432** (1994) 377-403

[33] A.J. Bracken, M.D. Gould and Y-Z. Zhang, 'From representations of the braid group to solutions of the Yang-Baxter equation', *Nucl. Phys.* **B354** (1991) 625-652